Eigenvalues & Eigenvectors

Conceptual knowledge

- Review eigenvalues, eigenvectors and eigenspaces.
- Understand how they relate to linear transformations.
- Be familiar with the various forms of matrix and how to obtain them.

Procedural knowledge

- Find the eigenvalues, eigenvectors and eigenspaces of a matrix.
- Determine what form a matrix is in.
- Diagonalise/reduce a matrix to Jordan normal form.
- Find a basis for the domain of a linear transformation whose matrix relative to that basis is diagonal.
- Find an orthogonal matrix that diagonalises a given matrix.

1 The Eigenvalue Problem

Eigenvectors and eigenvalues are defined by the eigenvalue problem for a given ***square matrix***, **A**.

$$\underline{A}\underline{x} = \lambda \underline{x}$$

Eigenvalue

Eigenvector

You can think about the eigenvalue problem as follows:

1. A system of linear equations is represented by a matrix.
2. There are certain vectors which come from the domain, and certain vectors which are part of the solution space.

3. The eigenvalue problem asks if there is a vector, $\mathbf{x}$, in the domain that we can put into the system and the solution vector is just a multiple of it.

The effect of the eigenvalue is to stretch or compress the vector, $\mathbf{x}$.

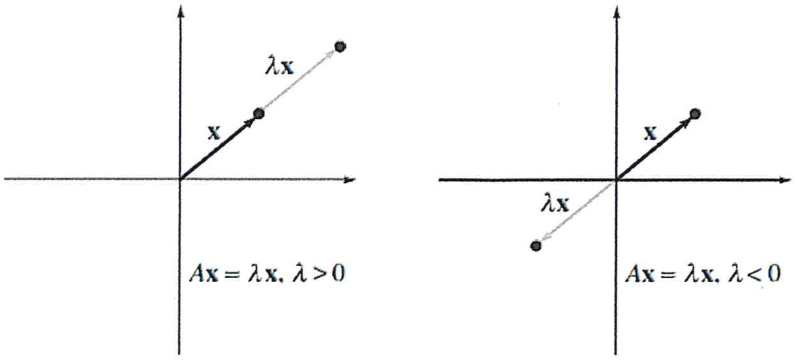


Figure 1: Effect of eigenvalue on eigenvector ©Houghton Mifflin Harcourt Publishing Company, 2009.

Note that by definition the zero vector can not be an eigenvector, however it is possible to have a 0 eigenvalue.

To verify eigenvectors for given eigenvalues, or eigenvalues for given eigenvectors, we simply need to substitute into the eigenvalue problem.

Example 1.1
Verify that $\mathbf{x} = (1, 0)$ *is an eigenvector of* $\mathbf{A}$ *and find its eigenvalue.*

Check $\underline{A}\underline{x} = \lambda \underline{x}$:

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

λ (eigenvalue)

$\underline{x}$

Union with 0 vector

eigenspace is Span$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ $(\lambda = 2)$ $\cup \{\underline{0}\}$

Definition

An ***eigenspace*** of a square matrix is the set of all non-zero eigenvectors corresponding with a particular eigenvalue, and the zero vector.

- $\underline{x} \neq \underline{0}$ (definition of eigenvector)
- Write: $\underline{\underline{B}} = (\underline{\underline{A}} - \lambda \underline{\underline{I}})$

$$\underline{\underline{B}}\underline{x} = (\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$$

$$\boxed{\underline{\underline{B}}\underline{x} = \underline{0}}$$ Homogeneous

- If $|\underline{\underline{B}}| \neq 0 \longrightarrow$ Unique solution $\rightarrow \underline{x} = \underline{0}$

Need $|\underline{\underline{B}}| = 0$ for ∞ infinite solutions

$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$$

Note that the definition above includes the zero vector so that it forms a subspace.

1.1 Finding Eigenvalues & Eigenvectors

To find the eigenvalues of a matrix we can solve the characteristic equation, which we derive as follows:

$$\underline{A}\underline{x} = \lambda \underline{x} \quad \text{(definition)}$$

$$\underline{A}\underline{x} = \lambda \underline{\underline{I}}\,\underline{x}$$

$$\underline{A}\underline{x} - \lambda \underline{\underline{I}}\,\underline{x} = \underline{0} \Rightarrow (\underline{A} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$$

$$\underline{x} \neq \underline{0}$$

The above homogeneous matrix equation must have a nontrivial solution since the eigenvector cannot be zero.

This happens when the matrix in the brackets is not invertible. In other words when it's determinant is 0.

> ***Definition***
>
> The ***characteristic equation*** of a matrix, $\mathbf{A}$, is given by,
>
> $$|\underline{A} - \lambda \underline{\underline{I}}| = 0$$

The solutions to this equation provide the eigenvalues.

Note that the resulting polynomial is called the ***characteristic polynomial***.

Example 1.2
Find the eigenvalues of,

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

$$\left| \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix} \right| = 0$$

$$(2-\lambda)(-6-\lambda) - 9 = 0$$

$$\lambda^2 + 4\lambda - 21 = 0$$

$$(\lambda - 3)(\lambda + 7) = 0 \Rightarrow \boxed{\lambda = 3, -7}$$

Once we have the eigenvalues we can find the corresponding eigenvectors by substituting the eigenvalue into the eigenvalue problem then solving the homogeneous system.

Example 1.3
Find the eigenvectors and dimensions of the eigenspaces for the previous example.

$\lambda = 3$: $\begin{pmatrix} 2-3 & 3 \\ 3 & -6-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{0}$

$\Rightarrow \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow x_1 = 3x_2$

Choose an x_2

Let, $x_2 = 1 \Rightarrow \underline{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector.

Check: $\underset{A}{\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}}\underset{\underline{x}}{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} = \begin{pmatrix} 9 \\ 3 \end{pmatrix} = \underbrace{3\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\lambda}$ ✓

Eigenspace: $E_{\lambda=3} = \{(3t, t) : t \in \mathbb{R}\} \cup \{\underline{0}\}$

Basis for $E_{\lambda=3} = \{(3,1)\}$ $\therefore$ $\boxed{\dim(E_{\lambda=3}) = 1}$

$\lambda = -7$: $\begin{pmatrix} 2+7 & 3 \\ 3 & -6+7 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \Rightarrow x_1 = -\frac{x_2}{3}$

Let, $x_2 = 3 \Rightarrow \underline{x} = (-1, 3)^T$ is an eigenvector.

Basis for $E_{\lambda=-7} = \{(-1,3)\}$

$\therefore$ $\boxed{\dim(E_{\lambda=-7}) = 1}$

Similarly to roots of polynomials, if an eigenvalue appears as a multiple root of the characteristic polynomial, we say it has ***multiplicity***.

A useful check is that the multiplicity of an eigenvalue is always greater than or equal to the dimension of its eigenspace.

$$\boxed{\text{The eigenvalues of a triangular } n \times n \text{ matrix are the entries on its main diagonal.}}$$

This result can easily be seen by considering the characteristic equation.

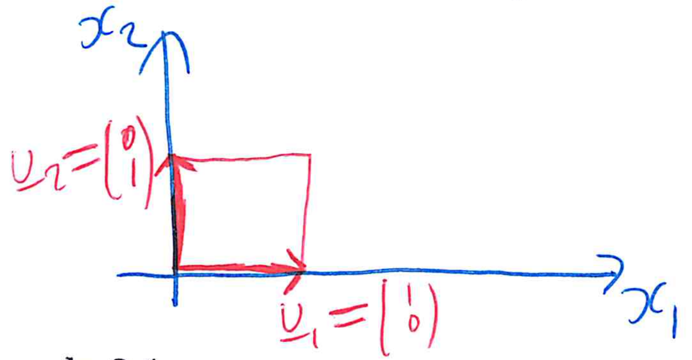
Be careful not to abuse this result - you cannot row reduce a matrix to triangular form then read off the eigenvalues because row operations change the matrix and its eigenvalues.

2 Geometric Significance of Eigenvalues & Eigenvectors

Whilst eigenvalues and eigenvectors play a crucial role in many areas of mathematics (for example when solving systems of nonlinear differential equations), they also have a geometric significance in $\mathbb{R}^3/\mathbb{R}^2$.

Since multiplying a vector by a matrix can be thought of as a linear transformation, let's see how various transformations affect the corresponding eigenvalues and eigenvectors.

In what follows we will consider a square defined by the standard basis vectors in $\mathbb{R}^2$.



Example 2.1
Investigate the eigenvalues/eigenvectors of the following matrix which represents a linear transformation.

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \qquad (k \neq 1) \in \mathbb{R}$$

Eigenvalues: $\lambda = k, 1$

Eigenvectors: $\lambda = k \Longrightarrow \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1-k \end{pmatrix}}_{\underline{A} - \lambda \underline{I}} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$x_1 \neq 0$

$\Longrightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector

$$\lambda = 1 \Rightarrow \begin{pmatrix} k-1 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; x_2 \neq 0$$

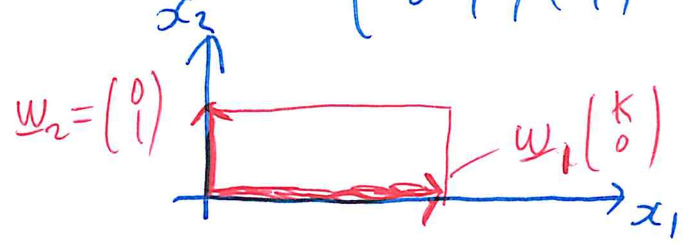
$$\Rightarrow \underline{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is an eigenvector}$$

In other words the eigenvectors are the same vectors we defined our square from.

Let's see how they are transformed:

$$\underline{w}_1 = \underline{\underline{A}}\,\underline{u}_1 = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{w}_2 = \underline{\underline{A}}\,\underline{u}_1 = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

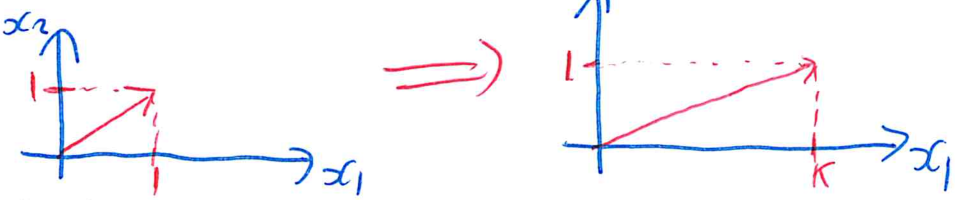


Horizontal stretch

We see that the eigenvectors have been stretched by a factor equal to their eigenvalue, however their direction has not been changed.

By comparison, let's see how a vector which is not an eigenvector gets transformed. Consider $\mathbf{u_3} = (1, 1)$:

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ 1 \end{pmatrix}$$



This vector has both been stretched and had its direction changed.

This result is true in general for linear transformations represented by matrices.

For a linear transformation represented by a matrix, the direction of its eigenvectors remains unchanged but they are stretched by a factor equal to their eigenvalue.

You will investigate more of these in the practice problems.

3 Diagonalisation

Diagonal matrices are used in some significant results from further linear algebra and computation.

Recall that 2 matrices, **A** and **B**, are similar if there exists an invertible, **P**, such that,

$$\underline{B} = \underline{P}^{-1}\underline{A}\,\underline{P}$$

> ***Definition***
>
> A square matrix, **A**, is diagonalisable if it is similar to a diagonal matrix.
>
> That is, there exists an invertible matrix, **P**, such that, $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

We develop 2 results which can help us with diagonal matrices.

> ***Theorem***
> If 2 square matrices are similar then they have the same eigenvalues.

Proof $|\underline{B} - \lambda \underline{I}| = |\underline{P}^{-1}\underline{A}\,\underline{P} - \lambda \underline{I}| = |\underline{P}^{-1}\underline{A}\,\underline{P} - \lambda \underline{I}\,\underline{P}\,\underline{P}^{-1}|$

$= |\underline{P}^{-1}\underline{A}\,\underline{P} - \underline{P}^{-1}\lambda \underline{I}\,\underline{P}|$ $\quad |\underline{A}\,\underline{B}| = |\underline{A}||\underline{B}| = |\underline{B}||\underline{A}|$

$= |\underline{P}^{-1}(\underline{A} - \lambda \underline{I})\underline{P}|$

$= |\underline{P}^{-1}||\underline{P}||\underline{A} - \lambda \underline{I}|$

$= |\underline{P}^{-1}\underline{P}||\underline{A} - \lambda \underline{I}| = |\underline{A} - \lambda \underline{I}|$ $\square$

> ***Theorem***
>
> An $n \times n$ matrix is diagonalisable if and only if it has n linearly independent eigenvectors.

Proof $\underline{\underline{A}}$ is diagonalisable $\Rightarrow \exists \boxed{\underline{\underline{D}} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}}$ column vectors

Let, $\underline{\underline{D}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{pmatrix}$, $\underline{\underline{P}} = (\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_n)$

$\boxed{\underline{\underline{P}}\,\underline{\underline{D}}} = (\lambda_1 \underline{P}_1 \; \lambda_2 \underline{P}_2, \ldots, \lambda_n \underline{P}_n) = \boxed{\underline{\underline{A}}\,\underline{\underline{P}}} = (\underline{\underline{A}}\underline{P}_1, \underline{\underline{A}}\underline{P}_2, \ldots, \underline{\underline{A}}\underline{P}_n)$

$\Rightarrow \boxed{\underline{\underline{A}}\,\underline{P}_i = \lambda_i \underline{P}_i, \; i \in [1, n]}$ $\square$

n Eigenvalue equations

We have just shown that if **A** is diagonalisable it has n eigenvectors.

We know that these are linearly independent because the eigenvectors are the columns of **P**, which is invertible.

We can repeat this argument backwards to prove the "if and only if" part of the theorem.

Example 3.1

The eigenvalues for the matrix, **A**, *are given below. Verify that the matrix whose columns are the eigenvectors produces a diagonal matrix through similarity.*

$(\lambda - 4)(\lambda + 2)^2 = 0$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \lambda_1 = 4, \quad \lambda_2 = -2 \text{ (multiplicity 2)}$$

Eigenvectors: $\lambda_1 = 4$: $\begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})$

$\lambda_2 = -2$: $\begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{x} = \begin{pmatrix} -x_2 \\ x_2 \\ x_3 \end{pmatrix}$

$= x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We have 3 linearly independent eigenvectors for a 3x3 matrix $\therefore \underline{\underline{A}}$ is diagonalisable.

$$\therefore \underline{P} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \underline{P}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{P}^{-1}\underline{\underline{A}}\,\underline{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{\underline{\underline{D}}}$$

Notice that the diagonal matrix has leading diagonal entries equal to the eigenvalues of **A**.

So if we can show that the matrix is diagonalisable then we can easily find the diagonal matrix to which it is similar by just substituting the eigenvalues.

Furthermore, we can check if a matrix is diagonalisable or not by checking to see if it has n linearly eigenvectors.

Example 3.2
Show that the following matrix is not diagonalisable.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$\underline{A}$ is triangular $\therefore \lambda = 1$ (multiplicity, 2)

$\lambda = 1$: $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow \underline{x} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Only 1 linearly independent eigenvector for 2x2 matrix

$\therefore$ $\underline{A}$ is not diagonalisable.

A useful result is that if an $n \times n$ matrix has n distinct eigenvalues then by definition there exist n linearly independent ~~eigenvalues~~ eigenvectors and therefore the matrix is diagonalisable.

Note that it is still possible for matrices with repeated eigenvalues to still be diagonalisable!

Example 3.3

Find a basis for the following linear transformation for which the corresponding matrix is diagonal.

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3)$$

Standard matrix: $\underline{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}$

Eigenvalues: $\left|\begin{pmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{pmatrix}\right| = (1-\lambda)\left[(3-\lambda)(-1-\lambda)-1\right] + \left[(-1-\lambda)+3\right] - \left[1+3(3-\lambda)\right]$

$= -\lambda^3 + 3\lambda^2 + 4\lambda - 12$

Factor theorem: $\lambda = 2$; $-8 + 12 + 8 - 12 = 0$ $\therefore (\lambda - 2)$ is a factor

$= (\lambda-2)(-\lambda^2+\lambda+6) = (\lambda-2)(\lambda+2)(-\lambda+3)$

$\therefore \boxed{\lambda = 2, -2, 3}$

Eigenvectors: $\lambda = 2$; $\begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \underline{x} = (-1, 0, 1)$ is an eigenvector.

$\lambda = -2$; $\underline{x} = (1, -1, 4)$

$\lambda = 3$; $\underline{x} = (-1, 1, 1)$

(Can do usual check for linear independence)

$\therefore$ diagonalisable.

$$
\begin{tikzcd}
\underline{v} \arrow[r, bend left, "T(\underline{v}) = \underline{\underline{A}}\,\underline{v}"] & T(\underline{v}) \arrow[d, "\underline{\underline{P}}^{-1}"] \\
{[\underline{v}]_B} \arrow[u, "\underline{\underline{P}}"] \arrow[r, bend right, "T([\underline{v}]_B) = \underline{\underline{D}}\,[\underline{v}]_B"'] & T([\underline{v}]_B)
\end{tikzcd}
$$

$$\underline{\underline{D}} = \underline{\underline{P}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{P}}^{*}$$

$$\underline{P} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{pmatrix}, \quad \underline{P}^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & 1 & \frac{1}{5} \end{pmatrix}$$

$$\underline{D} = \underline{P}^{-1}\underline{A}\,\underline{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

A basis for T with matrix, $\underline{D}$, is:

$$\{(-1,0,1), (1,-1,4), (-1,1,1)\}$$

3.1 Orthogonal Diagonalisation

Before we get to the final part of diagonalisation we will state some facts without proof.

The proofs will be guided in the practice problems for you to try.

A ***symmetric matrix*** is defined to be equal to its transpose and has the following properties.

1. Diagonalisable.
2. All eigenvalues are real.
3. Any eigenvalue with multiplicity, k, has k linearly independent eigenvectors (dimension of eigenspace is k).
4. Any 2 distinct eigenvalues have corresponding eigenvectors which are orthogonal.

An invertible ***orthogonal matrix*** has inverse equal to its transpose. $\underline{A}^{-1} = \underline{A}^T$

A general orthogonal matrix has columns which also form an orthonormal set.

Definition

A matrix, $\mathbf{A}$, is orthogonally diagonalisable if there exists an orthogonal matrix, $\mathbf{P}$, such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal.

The facts on the previous page lead us to the following fact.

> ***Theorem***
>
> A square matrix is orthogonally diagonalisable if and only if it is symmetric.

Proof

$\underline{A}$ is orthogonally diagonalisable $\Longrightarrow \exists\ \underline{D} = \underline{P}^{-1}\underline{A}\underline{P}$; $\underline{P}$ is orthogonal.

$\underline{P}^{-1} = \underline{P}^{T} \Longrightarrow \underline{A} = \underline{P}\,\underline{D}\,\underline{P}^{-1} = \underline{P}\,\underline{D}\,\underline{P}^{T}$

But, $\boxed{\underline{A}^T} = (\underline{P}\,\underline{D}\,\underline{P}^T)^T = \underline{P}\,\underline{D}^T\underline{P}^T = \underline{P}\,\underline{D}\,\underline{P}^{-1} = \boxed{\underline{A}}$

In the above we used the property of symmetric matrices that $\boxed{(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T.}$

The orthgonally diagonalise a symmetric matrix we perform the following steps:

1. Find eigenvalues and multiplicities.
2. For eigenvalues with multiplicity 1 get the eigenvector then normalise it.
3. For high multiplicity eigenvalues obtain the linearly independent eigenvectors then apply Gram-Schmidt orthonormalisation if necessary.

Example 3.4
Find an orthogonal matrix, $\mathbf{P}$*, that that orthogonally diagonalises the symmetric matrix,*

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$$

which has eigenvalues and eigenvectors,

$$\lambda_1 = -6, \mathbf{x_1} = (1,-2,2), \quad \lambda_2 = 3, \mathbf{x_2} = (2,1,0), \quad \lambda_3 = 3, \mathbf{x_3} = (-2,0,1)$$

$\lambda_1 = -6$; $\underline{v}_1 = \dfrac{\underline{x}_1}{\|\underline{x}_1\|} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$

$\lambda = 3$ (multiplicity, 2) ; Gram-Schmidt...

$$\underline{w}_2 = \underline{x}_2 = (2,1,0)\ ,\ \|\underline{w}_2\| = \sqrt{5}$$

$$\underline{w}_3 = (-2,0,1) - \overbrace{\frac{(-2,0,1)\cdot(2,1,0)}{\|(2,1,0)\|^2}(2,1,0)}^{\text{proj}_{\underline{w}_2}(-2,0,1)}$$

$$= \left(-\frac{2}{5}, \frac{4}{5}, 1\right)\ ,\ \|\underline{w}_3\| = \frac{3}{\sqrt{5}}$$

$$\Rightarrow \underline{v}_2 = \frac{\underline{w}_2}{\|\underline{w}_2\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$\Rightarrow \underline{v}_3 = \frac{\underline{w}_3}{\|\underline{w}_3\|} = \left(\frac{-2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}\right)$$

$$\therefore \underline{\underline{P}} = \begin{pmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{pmatrix}$$

You can verify that $\mathbf{P}^{-1}\mathbf{AP}$ gives a diagonal matrix with leading diagonal entries equal to the eigenvalues (remember the inverse of $\mathbf{P}$ is also its transpose).

3.2 Jordan Normal Form

Also known as ***Jordan canonical form*** is an almost diagonal matrix.

It's purpose is mostly for helping to obtain results in further linear algebra.

Given a square matrix, $\mathbf{A}$, it can be shown that we can always find an invertible matrix, $\mathbf{T}$, such that,

$$\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}} = \boxed{\underline{\underline{J}}}$$

where $\mathbf{J}$ is a ***block matrix*** (a matrix composed of other smaller matrices),

$$\underline{\underline{J}} = \begin{pmatrix} \underline{\underline{J}}_1 & & & \\ & \underline{\underline{J}}_2 & & \\ & & \ddots & \\ & & & \underline{\underline{J}}_n \end{pmatrix}$$

blocks

where each block, $\mathbf{J_i}$, is a square matrix of the form,

$$\underline{\underline{J}}_i = \lambda_i \underline{\underline{I}} + \underline{\underline{N}}_i$$

and $\mathbf{N}$ is a matrix with 1's on the diagonal line directly above the leading diagonal, with zeros everywhere else:

$$\underline{\underline{J}}_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & & \vdots \\ \vdots & 0 & \ddots & \ddots & 1 \\ \vdots & \vdots & & & \lambda_i \\ 0 & 0 & \cdots & & \end{pmatrix}$$

Each block has leading diagonal entries that correspond with an eigenvalue of the matrix.

Note that when a Jordan block is a 1×1 matrix then $\mathbf{N}$ is the zero matrix.

No 1's , $\underline{\underline{J}} = [\lambda]$

3.2.1 Finding the Jordan Normal Form

> ***Definition***
>
> A non-zero eigenvector of $\mathbf{A}$ is called a ***generalised eigenvector of rank*** r with associated eigenvalue, λ, if,
>
> $$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^r \underline{v} = \underline{0} \quad \text{and} \quad (\underline{\underline{A}} - \lambda \underline{\underline{I}})^{r-1} \underline{v} \neq \underline{0}$$

$r = 1 \Rightarrow$ regular eigenvector

> ***Theorem***
>
> An eigenvalue of multiplicity, m, has m linearly independent generalised eigenvectors.

Procedure:

1. Construct a linearly independent basis with a generalised eigenvector from each rank.

2. Find the Jordan blocks by constructing the matrix, $\mathbf{T}$, with columns given by the m linearly independent generalised eigenvectors.

For step 1 we start by choosing a generalised eigenvector ***of the highest rank***, N, then apply the recursive formula:

$$\boxed{\underline{x}_i = (\underline{A} - \lambda \underline{\underline{I}})^{N-i} \underline{x}_N, \quad i \in [1, N-1]}$$

Note that the ***Jordan normal form*** of a matrix is $\boxed{\textbf{\textit{unique}}}$ but the matrix used to obtain it, **T, *is not***.

↳ But allow blocks to be in any position

Example 3.5
Find the Jordan normal form of,

$$\mathbf{A} = \begin{pmatrix} -1 & -18 & -7 \\ 1 & -13 & -4 \\ -1 & 25 & 8 \end{pmatrix}$$

Eigenvalues: $|\underline{A} - \lambda \underline{\underline{I}}| \Longrightarrow (\lambda+2)^3 = 0$

$\therefore \lambda = -2$ (multiplicity, 3)

Rank 1: $(\underline{A} + 2\underline{\underline{I}}) = \begin{pmatrix} 1 & -18 & -7 \\ 1 & -11 & -4 \\ -1 & 25 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5/7 \\ 0 & 1 & 3/7 \\ 0 & 0 & 0 \end{pmatrix}$

$\Longrightarrow \underline{x} = (-5, -3, 7)$ is an eigenvector.

Rank 2: $(\underline{A} + 2\underline{\underline{I}})^2 = \begin{pmatrix} -10 & 5 & -5 \\ -6 & 3 & -3 \\ 14 & -7 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Longrightarrow \underline{x} = x_2(1, 2, 0) + x_3(1, 0, -2)$

3 eigenvectors. Linearly independent? No...

$$-\frac{2}{7}\begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix} - \frac{3}{7}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

Rank 3: $(\underline{A}+2\underline{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ← highest rank, $N=3$

Left blank for calculation

$$\Rightarrow \underline{x} = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

✱ Can choose any as long as they are linearly independent from rank 1 & 2 eigenvectors.

Choose, $\underset{\underline{x}_N}{\underline{x}_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\underline{x}_2 = (\underline{A}+2\underline{I})\underline{x}_3 = \begin{pmatrix} 1 & -18 & -7 \\ 1 & -11 & -4 \\ -1 & 25 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\underline{x}_1 = (\underline{A}+2\underline{I})^2\underline{x}_3 = \begin{pmatrix} -10 & 5 & -5 \\ -6 & 3 & -3 \\ 14 & -7 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ 14 \end{pmatrix}$$

$\underline{x}_1, \underline{x}_2, \underline{x}_3$ are linearly independent,

$$\Rightarrow \underline{T} = \begin{pmatrix} -10 & 1 & 1 \\ -6 & 1 & 0 \\ 14 & -1 & 0 \end{pmatrix}$$

Make sure correct order!

$$\underline{T}^{-1} = \begin{pmatrix} 0 & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{7}{4} & \frac{3}{4} \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \boxed{\underline{J} = \underline{T}^{-1}\underline{A}\,\underline{T} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{1 block}}$$