Practice Problems 6

1) Find the image of $\boldsymbol{v}$ and the preimage of $\boldsymbol{w}$.
a) $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2)$, $\boldsymbol{v} = (3, -4)$, $\boldsymbol{w} = (3,19)$
b) $T(v_1, v_2, v_3) = (v_2 - v_1, v_1 + v_2, 2v_1)$, $\boldsymbol{v} = (2,3,0)$, $\boldsymbol{w} = (-11, -1,10)$
c) $T(v_1, v_2, v_3) = (4v_2 - v_1, 4v_1 + 5v_2)$, $\boldsymbol{v} = (2, -3, -1)$, $\boldsymbol{w} = (3,9)$

2) Determine whether the following functions are linear transformations or not.
a) $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x, 1)$
b) $T: \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (x + y, x - y, z)$
c) $T: M_{33} \to M_{33}, T(\boldsymbol{A}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \boldsymbol{A}$
d) $T: \mathbb{R}^2 \to \mathbb{R}^3, T(x, y) = \left(\sqrt{x}, xy, \sqrt{y}\right)$

3) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (2,4, -1)$, $T(0,1,0) = (1,3, -2)$, and $T(0,0,1) = (0, -2,2)$. Find,
a) $T(0,3,1)$
b) $T(2, -4,1)$

4) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,1,1) = (2,0, -1)$, $T(0, -1,2) = (-3,2, -1)$, and $T(1,0,1) = (1,1,0)$. Find,
a) $T(2,1,0)$
b) $T(2, -1,1)$

5) Let $T: M_{22} \to M_{22}$ be a linear transformation such that

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \quad T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$$
$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$

Find $T\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$.

6) Find the kernel of the following linear transformations.
a) $T: \mathbb{R}^3 \to \mathbb{R}^3,\ T(x, y, z) = (0,0,0)$
b) $T: \mathbb{R}^3 \to \mathbb{R}^3,\ T(x, y, z) = (x, 0, z)$
c) $: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (x + 2y, y - x)$

7) Find a basis for the kernel and range of T, where the linear transformation is represented by $\boldsymbol{T(v) = Av}$.
a) $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
b) $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

c) $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{pmatrix}$

8) Find $ker(T)$, $nul(T)$, $range(T)$, $rank(T)$, where the linear transformation is represented by $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$.
a) $\boldsymbol{A} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$
b) $\boldsymbol{A} = \begin{pmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{pmatrix}$

9) Let $T: P_4 \to P_3$ be given by $T(p) = \frac{dp}{dx}$, where P_4 and P_3 are the vector spaces of polynomials of order 4 or less and 3 or less. What is the kernel of T?

10) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that projects the vector $\boldsymbol{u}$ onto $\boldsymbol{v} = (2,-1,1)$. Find the rank and nullity of T, and a basis for the kernel of T.

11) Find the standard matrices for the following linear transformations.
a) $T(x,y) = (x + 2y, x - 2y)$
b) $T(x,y) = (2x - 3y, x - y, y - 4x)$
c) $T(x,y,z) = (x + y, x - y, z - x)$

12) Find the standard matrices for the linear transformations and use it to find the image of $\boldsymbol{v}$. Describe and sketch the vector and its image.
a) $T(x,y) = (-x,-y)$, $\boldsymbol{v} = (3,4)$
b) $T(x,y) = (-x,y)$, $\boldsymbol{v} = (2,-3)$
c) $T(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, $\boldsymbol{v} = (4,4)$, $\theta = 135^o$

13) Find the standard matrix for the linear transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2$, that rotates a vector in $\mathbb{R}^2$ clockwise by θ degrees. Use it to find the image of $\boldsymbol{v} = (1,2)$ when $\theta = 60^o$.

14) Write down the standard matrices for the linear transformations that reflect through the xy-plane, xz-plane, and yz-plane.

15) Find the standard matrices for the linear transformations and use it to find the image of $\boldsymbol{v}$.
a) T is the projection onto the vector $\boldsymbol{w} = (3,1)$, $\boldsymbol{v} = (1,4)$.
b) T is the reflection through the vector $\boldsymbol{w} = (3,1)$, $\boldsymbol{v} = (1,4)$.

16) Find the standard matrices for $T = T_2 \circ T_1$.
a) $T_1(x,y) = (x - 2y, 2x + 3y)$, $T_2(x,y) = (2x, x - y)$
$(T_1, T_2: \mathbb{R}^2 \to \mathbb{R}^2)$
b) $T_1(x,y) = (-x + 2y, x + y, x - y)$, $T_2(x,y,z) = (x - 3y, z + 3x)$
$(T_1: \mathbb{R}^2 \to \mathbb{R}^3, T_2: \mathbb{R}^3 \to \mathbb{R}^2)$

17) Determine if the linear transformation is invertible or not. If it is, find the inverse transformation.

a) $T(x,y) = (x+y, x-y)$

b) $T(x,y) = (2x, 0)$

c) $T(x,y,z) = (x, x+y, x+y+z)$

d) $T(x,y) = (x+y, 3x+3y)$

18) Find the matrix of T relative to B and B'. Use it to find the transformation of the vector, $\boldsymbol{v}$ in B, with respect to the basis B'. (In other words the input vector is from the vector space with basis B, and the answer should be in the vector space with basis B').

a) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (x+y, x, y)$, $\boldsymbol{v} = (5,4)$

$B = \{(1,-1), (0,1)\}$, $\quad B' = \{(1,1,0), (0,1,1), (1,0,1)\}$

b) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x-y, y-z)$, $\boldsymbol{v} = (1,2,-3)$

$B = \{(1,1,1), (1,1,0), (0,1,1)\}$, $\quad B' = \{(1,2), (1,1)\}$

c) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (x+y+z, 2z-x, 2y-z)$, $\boldsymbol{v} = (4,-5,10)$

$B = \{(2,0,1), (0,2,1), (1,2,1)\}$, $\quad B' = \{(1,1,1), (1,1,0), (0,1,1)\}$

19) Find the matrix $\boldsymbol{A'}$, for T relative to the basis B' then show that it is similar to the standard matrix for $T, \boldsymbol{A}$.

a) $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x,y) = (2x-y, y-x), B' = \{(1,-2), (0,3)\}$

b) $T: \mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (x,y,z), B' = \{(1,1,0), (1,0,1), (0,1,1)\}$

c) $T: \mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (x-y+2z, 2x+y-z, x+2y+z), B' = \{(1,0,1), (0,2,2), (1,2,0)\}$

20) The linear transformation represented by the matrix, $\boldsymbol{A}$, is given with respect to B. Find the transformation of the vector, $[\boldsymbol{v}]_{B'}$ to the vector space with basis, B'.

a) $A = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}$, $[\boldsymbol{v}]_{B'} = (-1,2)$

$B = \{(1,3), (-2,-2)\}$, $\quad B' = \{(-12,0), (-4,4)\}$

b) $A = \begin{pmatrix} 3/2 & -1 & -1/2 \\ -1/2 & 2 & 1/2 \\ 1/2 & 1 & 5/2 \end{pmatrix}$, $[\boldsymbol{v}]_{B'} = (1,0,-1)$

$B = \{(1,1,0), (1,0,1), (0,1,1)\}$, $\quad B' = \{(1,0,0), (0,1,0), (0,0,1)\}$