Linear Transformations

Conceptual knowledge

- Understand the fundamentals of linear transformations and how they differ to other uses of the word "linear" you have encountered.

Procedural knowledge

- Find the image & preimage of a function.
- Find the kernal, range, bases for the kernel and range, rank and nullity of a linear transformation.
- Determine whether vector spaces are isomorphic or not.
- Find inverse transformations.
- Find the matrix of a linear transformation relative to arbitrary bases and utilise similar matrices.

1 Fundamentals

Linear transformations can be thought of as functions which ***map vectors*** from one vector space to another whilst ***preserving*** the addition and scalar multiplication ***operations***.

You will already be familiar with many of the fundamental ideas, but we will begin by formalising them in the context of linear transformations.

A function, T, ***maps*** elements of a vector space, V, to elements in a vector space, W, and is denoted by, $T: V \to W$

V is the ***domain*** and W is the ***codomain***. "All possible output vectors"

Consider the input and output of this function:

$$T(\underline{v}) = \underline{w} \leftarrow \text{Image}$$

We call $\mathbf{w}$ the ***image*** of $\mathbf{v}$ under T.

The set of all images of vectors in V is called the ***range***.

The set of all $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$ is called the ***preimage*** of $\mathbf{w}$.

Example 1.1

Find the image of $\mathbf{v} = (-1, 2)$ *and the preimage of* $\mathbf{w} = (-1, 11)$ *for the function mapping* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *given by,*

$$T(\mathbf{v}) = (u_1 - v_2, v_1 + 2v_2)$$

Image: $T(-1,2) = (-1-2, -1+4) = (-3,3)$

Pre-image: $(v_1 - v_2, v_1 + 2v_2) = (-1, 11)$

$$\begin{aligned} v_1 - v_2 &= -1 \\ v_1 + 2v_2 &= 11 \end{aligned} \Rightarrow \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 2 & 11 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 3 & 12 \end{array}\right)$$

$\Rightarrow v_2 = 4$, $v_1 = -1 + 4 = 3$ $\therefore$ pre-image is $(3,4)$

Definition

Let V and W be vector spaces. The function $T : V \to W$ is called a ***linear transformation*** of V into W if for all $\mathbf{u}$ and $\mathbf{v}$ in V, and any scalar, c,

1. $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$
2. $T(c\underline{u}) = cT(\underline{u})$

Example 1.2

Show that the function in Example 1.1 is a linear transformation.

1) $T(\underline{u} + \underline{v}) = ([u_1 + v_1] - [u_2 + v_2], [u_1 + v_1] + 2[u_2 + v_2])$

$= ([u_1 - u_2] + [v_1 - v_2], [u_1 + 2u_2] + [v_1 + 2v_2])$

$$= (v_1 - v_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2)$$

$$= T(\underline{u}) + T(\underline{v})$$

2) At home...

Example 1.3
Show that the functions $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = x + 1$ are not linear transformations.

$$f(x) = \sin(x) \rightarrow f(x_1 + x_2) = \sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2) = f(x_1) + f(x_2)$$

$$g(x) = x^2 \rightarrow g(cx) = (cx)^2 = c^2x^2 \neq cx^2 = cg(x)$$

$$h(x) = x + 1 \rightarrow h(cx) = cx + 1 \neq c(x+1) = cx + c = ch(x)$$

Note that even some linear graphs (straight line) are not classed as a linear transformation since if they do not preserve addition and scalar multiplication.

We also have two fundamental linear transformations known as the ***zero transformation*** and the ***identity transformation***:

$$T(\underline{v}) = \underline{0} \quad , \quad T(\underline{v}) = \underline{v}$$

An interesting feature of linear transformations is that the zero vector is always mapped to itself:

$$T(\underline{0}) = \underline{0} \quad \text{(always true)}$$

because,

$$T(\underline{0}) = T(0\underline{v}) = 0T(\underline{v}) = \underline{0} \quad \forall \underline{v} \in V$$

2 Linear Transformations, Bases & Matrices

If we know how a linear transformation maps a basis we know how it maps the entire vector space since the basis spans that vector space.

Example 2.1

A transformation, $T : \mathbb{R}^3 \to \mathbb{R}^3$, maps the standard basis as follows. Use the information to find $T(2,3,-2)$.

$$T(1,0,0) = (2,-1,4), \qquad T(0,1,0) = (1,5,-2), \qquad T(0,0,1) = (0,3,1)$$

$(2,3,-2) = 2(1,0,0) + 3(0,1,0), -2(0,0,1)$

$\therefore T(2,3,-2) = T\big(2(1,0,0) + 3(0,1,0) - 2(0,0,1)\big)$

$= 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1)$

$= 2(2,-1,4) + 3(1,5,-2) - 2(0,3,1)$

$= (7,7,0)$

We know from previous lectures that matrices formed from vectors in a basis provides us with significant tools to analyse linear systems.

In fact we have that a general $m \times n$ matrix always represents a linear transformation.

> ***Theorem***
>
> Let $\mathbf{A}$ be an $m \times n$ matrix. The function, $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$, is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

You are asked in the practice problems to prove the theorem above. You can do this by just writing out the multiplication and checking the 2 axioms that define a linear transformation.

Note that the transformation can be between vector spaces of different dimensions.

Example 2.2

The matrices below represent linear transformations. Find the dimensions of the domain and codomain.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{pmatrix}$$

$\underline{\underline{A}}\underline{v} = \underline{w}$: $\begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}}_{\text{Input}} = \underbrace{\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}}_{\text{output}}$, $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$

$\underline{\underline{B}}\underline{v} = \underline{w}$: $\begin{pmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$

2.1 Kernel

Definition

For a linear transformation, $T : V \to W$, the ***kernel*** is defined as,

$$\mathrm{Ker}(T) = \{ \underline{v} \in V : T(\underline{v}) = \underline{0} \}$$

Note: $\underline{0}$ is always in the kernel : $T(\underline{0}) = \underline{0}$

In other words it is the preimage of the zero vector.

Example 2.3

Find the kernel of the transformation, $T(x, y, z) = (x, y, 0)$. $= (0,0,0)$?

$(x, y, 0) \to (0,0,0)$: $\mathrm{Ker}(T) = \{ (0,0,z) : z \in \mathbb{R} \}$

Example 2.4

Find the kernel of the linear transformation represented by the following matrix.

$T(\underline{v}) = \underline{\underline{A}}\underline{v}$

$$\underline{\underline{A}} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix}, \quad T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$\begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow v_1 = v_3, \quad v_2 = -v_3$$

$$\therefore \ker(T) = \{(t, -t, t) : t \in \mathbb{R}\}$$ same

$$= \text{Span}\{(1, -1, 1)\}$$

We basically have that any linear transformation represented by a matrix has a kernel equal to the nullspace of the matrix.

Don't forget that not all linear transformations are matrices.

Example 2.5
Show that the definite integral operator is a linear transformation, $T : C[a, b] \to \mathbb{R}$.

$$T(f) = \int_a^b f(x)\,dx$$

$$T(f+g) = \int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = T(f) + T(g)$$

$$T(cf) = \int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx = c\,T(f)$$

2.2 Range of a Linear Transformation

In a previous lecture we saw that for the matrix equation, $\mathbf{Ax} = \mathbf{b}$, the column space of $\mathbf{A}$ gave all vectors $\mathbf{b}$ that satisfy the equation. range

That means that linear transformations represented by matrices have range equal to the column space.

We then have the following definitions.

> ***Definition***
>
> For a linear transformation, $T : V \to W$, the ***nullity*** of T is the dimension of the kernel.
>
> The ***rank*** of T is the dimension of the range. $\rightarrow \dim(\text{col } \underline{A})$

$\underline{\underline{A}}\,\underline{x} = \underline{b}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$x_1\begin{pmatrix} a \\ c \end{pmatrix} + x_2\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Linear combination of

$\mathrm{Col}(\underline{\underline{A}})$ = all possible $\underline{b}$ (range)

Example 2.6

Find the rank and nullity of the linear transformation represented by the following matrix (reduced echelon form given).

$T: \mathbb{R}^5 \to \mathbb{R}^4$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\underline{A}\underline{x} = \underline{b}$ Particular solution

$Col(\underline{A}) = Span\{(1,2,-1,0),(2,1,0,0),(1,1,0,2)\}$

$dim(Col(\underline{A})) = 3$ $\boxed{\therefore rank(T) = 3}$

$\Rightarrow v_4 = -4v_5,\ v_2 = v_3 + 2v_5,\ v_1 = -2v_3 + v_5$

$$\underline{v} = \begin{pmatrix} -2v_3 + v_5 \\ v_3 + 2v_5 \\ v_3 \\ -4v_5 \\ v_5 \end{pmatrix} = v_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_5 \begin{pmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

Homogeneous solution

basis for the null space $\underline{A}\underline{x} = 0$

$\boxed{\therefore Nul(T) = 2}$

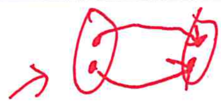
Note that the sum of the rank and nullity gives the dimension of the domain.

2.3 Isomorphisms

The prefix "iso" can generally be thought of as "the same" or "equal".

Isomorphic vector spaces basically implies that the vector spaces are really the same thing but perhaps look a little different.

Before we define isomorphism let's extend the definitions of one-to-one and onto to linear transformations.

> ***Definition***
>
> 
>
> A linear transformation, $T : V \to W$, is ***one-to-one*** (or injective) if the preimage of every vector in the range consists of a single vector.
>
> It is ***onto*** (or surjective) if every element of W has a preimage in V.

In other words, for onto transformations the codomain equals the range. (onto)

Test for one-to-one

$$\ker(T) = \underline{0}$$

Test for onto

$$\underbrace{\text{Rank}(T)}_{\dim(\text{col}(\underline{A}))} = \dim(W) \quad \text{codomain}$$

> ***Definition***
>
> A linear transformation, $T : V \to W$, is called an ***isomorphism*** if it is both on-to-one and onto.
>
> The vector spaces V and W are ***isomorphic*** to each other.

We have that 2 vectors are ***isomorphic*** if and only if their ***dimensions are equal***. Remember we have,

$$\text{rank}(T) + \text{nul}(T) = \dim(\text{domain})$$

so isomorphic vector spaces imply:

One-to-one: $\text{Nul}(\underline{A}) = \ker(T) = \underline{0} \longrightarrow \text{dimension} = 0$

$\text{Rank}(T) = \dim(\text{domain})$

$\dim(\text{range}) = \dim(\text{domain})$] equal

Onto: $\dim(\text{range}) = \dim(\text{codomain})$

$\boxed{\dim(\text{domain}) = \dim(\text{codomain})}$ Test for isomorphism

Example 2.7

Show that $\mathbb{R}^4$ and M_{22} are isomorphic to each other.

$$\text{Basis for } \mathbb{R}^4 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\} \quad \dim =$$

$$\text{Basis for } M_{22} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad \dim =$$

If $T(\underline{u}) = T(\underline{v})$

$$T(\underline{u}) - T(\underline{v}) = \underline{0}$$

$$T(\underline{u} - \underline{v}) = \underline{0}$$

$\boxed{\text{If } \ker(T) = \underline{0}} \longrightarrow \underline{u} - \underline{v} = \underline{0}$

$\therefore \underline{u} = \underline{v}$ $\square$

3 Linear Transformation Matrices & Similarity

Since we have seen that any vector spaces with the same dimensions are simply isomorphic, we will now focus on results for $\mathbb{R}^n$, happy in the knowledge we can apply them to other vector spaces (such as continuous functions etc.).

> ***Definition***
>
> The ***standard matrix***, $\mathbf{A}$, of a linear transformation, $T : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix whose columns are the transformations of the standard basis vectors which results in,
>
> $$T(\underline{v}) = \underline{\underline{A}}\,\underline{v}$$

So for $\mathbb{R}^2$ we have, $B = \{(1,0), (0,1)\}$

$$T(1,0) = (a,b), \quad T(0,1) = (c,d)$$

Standard matrix! $\underline{\underline{A}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

> ***Definition***
>
> The ***composition***, T, of two linear transformations, $T_1 : \mathbb{R}^n \to \mathbb{R}^m$, and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$, is defined as,
>
> $$T(\underline{v}) = T_2(T_1(\underline{v})) = (T_2 \circ T_1)\underline{v}$$

If the associated standard matrices for T, T_1 and T_2 are $\mathbf{A}$, $\mathbf{A_1}$ and $\mathbf{A_2}$ respectively then we have,

$$\underline{\underline{A}}\,\underline{v} = \underline{\underline{A}}_2\,\underline{\underline{A}}_1\,\underline{v}$$

Example 3.1

Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$, for,

$$T_1(x, y, z) = (2x + y, 0, x + z), \qquad T_2(x, y, z) = (x - y, z, y)$$

$$\underline{\underline{A}}_1 = \left(T_1(1,0,0),\ T_1(0,1,0),\ T_1(0,0,1)\right) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{A}}_2 = \left(T_2(1,0,0),\ T_2(0,1,0),\ T_2(0,0,1)\right) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T = \underline{\underline{A}}\,\underline{v} = (T_2 \circ T_1)\underline{v} = \underline{\underline{A}}_2 \underline{\underline{A}}_1 \underline{v} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T' = \underline{\underline{A}}'\underline{v} = (T_1 \circ T_2)\underline{v} = \underline{\underline{A}}_1 \underline{\underline{A}}_2 \underline{v} = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that the linear transformations represented by matrices are ***invertible*** if their standard matrix is invertible.

Example 3.2

Find the inverse of the linear transformation given by,

$$T(x, y, z) = (2x + 3y + z, 3x + 3y + z, 2x + 4y + z)$$

Standard matrix: $\underline{\underline{A}} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{pmatrix}$, $|\underline{\underline{A}}| = 1$

$$\therefore \underline{\underline{A}}^{-1} = \frac{1}{1}\begin{pmatrix} \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix}$$

$$T^{-1}(x,y,z) = (-x+y, -x+z, 6x-2y-3z)$$

3.1 Non-standard Bases

Earlier we saw how to transform coordinates from one basis to another using transition matrices.

We will now see how to get the coordinates of a linear transformation from one basis to another.

Say we have a linear transformation, $T : V \to W$, where B is a basis for V and B' is a basis for W.

$$T(\underline{v}) = \underline{\underline{A}}\,\underline{v} = \underline{w} \quad \text{(standard basis)}$$

$$\begin{tikzcd}
\underline{v} \arrow[r] \arrow[rrr, bend right, "\underline{\underline{A}}\,\underline{v}"'] & {[\underline{v}]_B} \arrow[r] & \boxed{\underline{\underline{A}}^*[\underline{v}]_B = [\underline{w}]_{B'}} \arrow[r] & \underline{w}
\end{tikzcd}$$

$\underline{v}$ is standard basis

So we want to find a matrix, $\mathbf{A}^*$, that allows us to skip having to re-write the vector in terms of the other basis before the linear transformation.

We call $\mathbf{A}^*$ the matrix for T relative to B and B'.

$$\underline{\underline{P}}[\underline{v}]_B = \underline{v} = \underline{\underline{Q}}[\underline{v}]_{B'} \Rightarrow [\underline{v}]_{B'} = \underline{\underline{Q}}^{-1}\underline{\underline{P}}[\underline{v}]_B$$

$$\underline{\underline{A}}^*[\underline{v}]_B = [\underline{w}]_{B'}, \quad \underline{w} = \underline{\underline{Q}}[\underline{w}]_{B'} = \text{[redacted]} = \underline{\underline{Q}}\,\underline{\underline{A}}^*[\underline{v}]_B$$

$$= \underline{\underline{A}}\,\underline{v} = \underline{\underline{A}}\,\underline{\underline{P}}[\underline{v}]_B \Rightarrow \underline{\underline{Q}}\,\underline{\underline{A}}^* = \underline{\underline{A}}\,\underline{\underline{P}}$$

> The matrix $\mathbf{A}^*$ is just the original transformation matrix multiplied with the transition matrices from B to B'.

(*) We could also achieve the same result by transforming the basis vectors in B, then writing them with respect to the vectors in B'.

$$\boxed{\underline{\underline{A}}^* = \underline{\underline{Q}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}}$$

When $B = B'$ we call $\mathbf{A}$ the matrix of T relative to B.

Example 3.3
Find the transformation matrix of T relative to B and B' then apply it to the vector $[\mathbf{x}]_B = (3, 4)$.

$$T(x_1, x_2) = (x_1 + x_2, 2x_2 - x_2), \qquad B = (1, 2), (-1, 1), \qquad B' = (1, 0), (0, 1)$$

$A^* = Q^{-1} A P$, $P = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\therefore Q^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= AP = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$

$A^*[x]_B = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ -12 \end{pmatrix} = [w]_{B'} = w$ $(B' = S)$

Check: $v = P[v]_B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \end{pmatrix}$ standard basis

$Av = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 10 \end{pmatrix} = \begin{pmatrix} 9 \\ -12 \end{pmatrix} = w$ ✓

Alternatively: $T(1,2) = (3,0)$, $T(-1,1) = (0,-3)$

B': $(3,0) = 3(1,0) + 0(0,1)$, $(0,-3) = 0(1,0) - 3(0,1)$

$\therefore A^* = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$

3.2 Similarity

Consider,

1. $\mathbf{A}$ is the matrix of T relative to B.
2. $\mathbf{A}'$ is the matrix of T relative to B'.
3. $\mathbf{R}$ is transition matrix, $B' \to B$.
4. $\mathbf{R}^{-1}$ is transition matrix, $B \to B'$.

We can use the following definition to transformation matrices relative to arbitrary bases.

The result is also used in later work.

> ***Definition***
>
> Two square matrices of the same size, $\mathbf{A}$ and $\mathbf{A}'$, are ***similar*** if there exists an invertible matrix, $\mathbf{R}$, such that
>
> $$A' = R^{-1} A R.$$

$$
\begin{tikzcd}
{[v]_B} \arrow[r, "A"] & T([v]_B) \arrow[d, "R^{-1}"] \\
{[v]_{B'}} \arrow[u, "R"] \arrow[r, "A'"] & T([v]_{B'})
\end{tikzcd}
$$

Example 3.4
The following linear transformation is given relative to the standard basis. Find the matrix of T relative to B'.

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2), \qquad B' = (1,0), (1,1)$$

$$P[v]_S = v = Q[v]_{B'} \qquad P = I, \quad Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = R$$

$$R^{-1} = Q^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\therefore\ A' = R^{-1} A R = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$$

Test: $[v]_{B'} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \longrightarrow v = \ldots \quad T(v) = \ldots \longrightarrow [T(v)]_{B'}$

Compare with: $A'[v]_{B'} = \ldots$?