Practice Problems 4

1) Find the length of the vectors.
a) $\mathbf{v} = (4,3)$
b) $\mathbf{v} = (1,2,2)$
c) $\mathbf{v} = (2,0,-5,5)$

2) Find $||\mathbf{u}||, ||\mathbf{v}||$ and $||\mathbf{u}+\mathbf{v}||$.
a) $\mathbf{u} = (0,4,3), \mathbf{v} = (1,-2,1)$
b) $\mathbf{u} = (0,1,-1,2), \mathbf{v} = (1,1,3,0)$

3) Find a unit vector in the direction of $\mathbf{u}$.
a) $\mathbf{u} = (-5,12)$
b) $\mathbf{u} = (1,0,2,2)$

4) For what values of c is $||c(1,2,3)|| = 7$?

5) Find the vector, $\mathbf{v}$, with the specified length in the same direction as $\mathbf{u}$.
a) $||\mathbf{v}|| = 2$, $\mathbf{u} = (\sqrt{3},3,0)$
b) $||\mathbf{v}|| = 3$, $\mathbf{u} = (0,2,1,-2)$

6) Find the distance between $\mathbf{u}$ and $\mathbf{v}$.
a) $\mathbf{u} = (1,1,2)$, $\mathbf{v} = (-1,3,0)$
b) $\mathbf{u} = (0,1,2,3)$, $\mathbf{v} = (1,0,4,-1)$

7) Find $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{u}$, $||\mathbf{u}||$, and $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$.
$\mathbf{u} = (-1,1,2)$, $\mathbf{v} = (1,-3,-2)$

8) Find $(\mathbf{u}+\mathbf{v})\cdot(2\mathbf{u}-\mathbf{v})$ given that $\mathbf{u}\cdot\mathbf{u} = 4$, $\mathbf{u}\cdot\mathbf{v} = -5$ and $\mathbf{v}\cdot\mathbf{v} = 10$.

9) Find the angle between the vectors.
a) $\mathbf{u} = (1,1,1)$, $\mathbf{v} = (2,1,-1)$
a) $\mathbf{u} = (0,1,0,1)$, $\mathbf{v} = (3,3,3,3)$
a) $\mathbf{u} = (1,3,-1,2,0)$, $\mathbf{v} = (-1,4,5,-3,2)$

10) Determine all vectors orthogonal to $\mathbf{u}$.
a) $\mathbf{u} = (0,5)$
b) $\mathbf{u} = (2,-1,1)$

11) Find the angle between the diagonal of a cube and the diagonal of one of its sides.

12) Prove that if $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{R}^n$ then $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$.

13) Prove that if $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$ then $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$.

14) Find $\langle \mathbf{u}, \mathbf{v} \rangle$, $||\mathbf{u}||, ||\mathbf{v}||$ and $d(\mathbf{u}, \mathbf{v})$ for the specified inner products.
a) $\mathbf{u} = (3,4)$, $\mathbf{v} = (5,-12)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
b) $\mathbf{u} = (-4,3)$, $\mathbf{v} = (0,5)$, $\langle \mathbf{u}, \mathbf{v} \rangle = 3\mathrm{u}_1 v_1 + u_2 v_2$
c) $\mathbf{u} = (8,0,-8)$, $\mathbf{v} = (8,3,16)$, $\langle \mathbf{u}, \mathbf{v} \rangle = 2\mathrm{u}_1 v_1 + 3u_2 v_2 + u_3 v_3$

15) Find $\langle f, g \rangle$, $||f||, ||g||$ and $d(f, g)$ for the inner product,
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx$$
a) $f(x) = x^2$, $g(x) = x^2 + 1$
b) $f(x) = x$, $g(x) = e^x$

16) Find $\langle \mathbf{A}, \mathbf{B} \rangle$, $||\mathbf{A}||, ||\mathbf{B}||$ and $d(\mathbf{A}, \mathbf{B})$ for the inner product,
$$\langle \mathbf{A}, \mathbf{B} \rangle = 2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22}$$
a) $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$
b) $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

17) Find $\langle p, q \rangle$, $||p||, ||q||$ and $d(p, q)$ for the inner product,
$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$
a) $p(x) = 1 - x + 3x^2$, $q(x) = x - x^2$
b) $p(x) = 1 + x^2$, $q(x) = 1 - x^2$

18) Prove that
$$\langle \mathbf{A}, \mathbf{B} \rangle = 2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22}$$
is an inner product.

19) Find the angle between the vectors.
a) $\mathbf{u} = (-4,3), \mathbf{v} = (0,5)$, $\langle \mathbf{u}, \mathbf{v} \rangle = 3\mathrm{u}_1 v_1 + u_2 v_2$
b) $\mathbf{u} = (1,1,1), \mathbf{v} = (2,-2,2)$, $\langle \mathbf{u}, \mathbf{v} \rangle = 2\mathrm{u}_1 v_1 + u_2 v_2 + u_3 v_3$
c) $p(x) = 1 - x + x^2$, $q(x) = 1 + x + x^2$, $\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$
d) $f(x) = x$, $g(x) = x^2, \langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx$

20) Verify the Cauchy-Schwarz inequality and the triangle inequality.
a) $\mathbf{u} = (1,0,4)$, $\mathbf{v} = (-5,4,1)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
b) $p(x) = 2x$, $q(x) = 3x^2 + 1$, $\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$
c) $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -3 & 1 \\ 4 & 3 \end{pmatrix}$,
$\langle \mathbf{A}, \mathbf{B} \rangle = 2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + 2a_{22}b_{22}$
d) $f(x) = \sin x, g(x) = \cos x$,
$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx$$

21) Show that f and g are orthogonal for the inner product,

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$

on $C[a,b]$.

a) $C[-\pi,\pi]$, $f(x) = \cos x$, $g(x) = \sin x$

b) $C[-1,1]$, $f(x) = x$, $g(x) = \frac{1}{2}(5x^3 - 3x)$

22) Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ or $\text{proj}_g f$ on $C[a,b]$.

a) $\mathbf{u} = (1,3,-2)$, $\mathbf{v} = (0,-1,1)$, $\langle \mathbf{u},\mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v}$

b) $\mathbf{u} = (0,1,3,-6)$, $\mathbf{v} = (-1,1,2,2)$, $\langle \mathbf{u},\mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v}$

c) $C[0,1]$, $f(x) = x$, $g(x) = e^x$, $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$

d) $C[-\pi,\pi]$, $f(x) = \sin x$, $g(x) = \cos x$, $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$

e) $C[-\pi,\pi]$, $f(x) = x$, $g(x) = \sin 2x$, $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$

23) a) Prove that if $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{R}^n$ then $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{u},\mathbf{w}\rangle$.

b) Prove that if $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$ with $c \in \mathbb{R}$ then $\langle \mathbf{u}, c\mathbf{v}\rangle = c\langle \mathbf{u},\mathbf{v}\rangle$.

c) Let W be a subspace of the inner product space, V. Show that Q is a subspace of V.

$$Q = \{\mathbf{v} \in V : \langle \mathbf{v},\mathbf{w}\rangle = 0\ \forall \mathbf{w} \in W\}$$