Vector Spaces (Part II)

Conceptual knowledge

- Understand the meaning of dimension, rank and change of basis.
- Understand general coordinate systems using vectors.

Procedural knowledge

- Find the dimension, rank and bases of a vector space.
- Find coordinates of vectors relative to a basis.
- Find transition matrices between different bases.

1 Dimension, Basis & Coordinates

Last time we saw that if a basis for a vector space has n vectors in it, then every other basis also has n vectors.

The number, n, is called the ***dimension*** of the vector space, V, denoted by $dim(V)$.

Example 1.1
Find the dimensions of the the vector spaces, $\mathbb{R}^3$, P_3 (polynomials of degree 3 or less), and M_{23} (2×3 matrices).

Standard basis for $\mathbb{R}^3$: $\{(1,0,0), (0,1,0), (0,0,1)\}$

$\Rightarrow dim(\mathbb{R}^3) = 3$

Standard basis for P_3: $\{1, x, x^2, x^3\}$

$\Rightarrow dim(P_3) = 4$

Standard basis for M_{23}: $\left\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\right\}$

$\Rightarrow dim(M_{23}) = 6$

In general we have that $dim(\mathbb{R}^n) = n$, $dim(P_n) = n + 1$, and $dim(M_{mn}) = m \times n$.

Geometrically for the vector space, $\mathbb{R}^3$, you can imagine the ***standard basis vectors*** as the unit vectors along the ***coordinate axes***.

A ***nonstandard*** basis would simply be an ***alternative coordinate*** system with axes defined by the basis vectors.

This is where the intuitive notion of "dimension" comes in, however it can now be thought of more generally in the context of vector spaces.

To determine the dimension of a vector space, in general we must find a basis.

1.1 Finding a Basis

Finding a basis involves ***finding a spanning set***, then making sure it is ***linearly independent***.

We already know how to check for linear independence, but ***what happens if*** the spanning set we find is ***linearly dependent***?

The answer is:

> We discard unnecessary vectors until we have a linearly independent set, whilst making sure the set still spans the vector space.

> ***Theorem (Spanning Set)***
>
> Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$ be a set in a vector space, V, and let $H = Span\, S$.
>
> 1. If a vector in S, $\mathbf{v_k}$, is a linear combination of the remaining vectors, then the set formed from S by removing $\mathbf{v_k}$ still spans H.
>
> 2. If $H \neq \{\mathbf{0}\}$ then some subset of S is a basis for H. Note: $\dim(\{\underline{0}\}) = 0$

Proof

1) • Re-arrange such that $\underline{v}_p = \underline{v}_k$

• Let $\underline{v}_p = a_1\underline{v}_1 + a_2\underline{v}_2 + \ldots + a_{p-1}\underline{v}_{p-1}$

• For any $\underline{x} \in H$:

$$\underline{x} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_{p-1}\underline{v}_{p-1} + c_p\underline{v}_p$$

$$\therefore \underline{x} = (a_1+c_1)\underline{v}_1 + (a_2+c_2)\underline{v}_2 + \dots + (c_{p-1}+a_{p-1})\underline{v}_{p-1}$$

$$\therefore \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{p-1}\} \text{ spans } H.$$

2) Omitted

Example 1.2
Find a basis for the vector space, V, spanned by $\mathbf{v_1} = (1,1,0)$, $\mathbf{v_2} = (0,1,1)$, $\mathbf{v_3} = (2,3,1)$, *and* $\mathbf{v_4} = (1,1,1)$.

$$V = \text{span}\{(1,1,0),(0,1,1),(2,3,1),(1,1,1)\}$$

Make linearly independent by removing vectors.

$$a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 + a_4\underline{v}_4 = \underline{0} \; ?$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Reduced echelon form

$$\Rightarrow a_4 = 0, \quad a_2 = -a_3, \quad a_1 = -2a_3$$

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -2a_3 \\ -a_3 \\ a_3 \\ 0 \end{pmatrix} = a_3\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

Taking $a_3 = -1$ $\therefore$ $2\underline{v}_1 + \underline{v}_2 - \underline{v}_3 = \underline{0}$

These vectors are linearly dependent, therefore we can drop any of them.

The convention is to keep ones corresponding with the pivot positions.

Basis: $\text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_4\} = \text{Span}\{(1,1,0), (0,1,1), (1,1,1)\}$

?

Let us check whether the new spanning set is linearly independent.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \Rightarrow \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore$ invertible $\Rightarrow$ Unique solution, $\because$ homogeneous must be trivial solution.

$\therefore$ Set is linearly independent, $\therefore$ a basis.

Let us just recap some key points from the last example:

- To find a basis we take a spanning set and check for linear independence.
- If it is linearly dependent we remove a vector and check again (repeat as necessary).
- Adding a multiple of one row to another leaves the determinant unchanged (swapping rows swaps the sign of the determinant and multiplying a row by a constant scales the determinant).
- Non-zero determinant means unique solution.
- For homogeneous systems a unique solution means the trivial solution.

Example 1.3

Solution set to $\underline{A}\underline{x} = \underline{0}$

Find a basis, and hence the dimension, of the null space of the following system.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Longrightarrow x_6 = 0$, x_2, x_4, x_5 are free.

$\Longrightarrow$ $x_3 = -2x_4$, $x_1 = -3x_2 - 4x_4 - 2x_5$

$$\therefore \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3x_2 - 4x_4 - 2x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \\ 0 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\underline{v}_1} + x_4 \underbrace{\begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\underline{v}_2} + x_5 \underbrace{\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\underline{v}_3}$$

$\therefore \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ Spans null space.

So we have found a spanning set for the null space. Let us check for linear independence.

$$\left(\begin{array}{ccc|c} -3 & -4 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & -\frac{4}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \text{Trivial Solution}$$

$\therefore$ Linearly independent.

So the spanning set is linearly independent and therefore $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for the null space.

This means that the dimension of the null space is 3.

1.2 Finding the Dimension of Row/Column Spaces

When we perform elementary row operations on a matrix then the row space is unchanged. (But column space changes)

To see this consider a matrix, $\mathbf{A}$, which is row reduced to $\mathbf{B}$.

The rows of $\mathbf{B}$ are just a linear combination of the rows in $\mathbf{A}$.

This means that the rows of $\mathbf{B}$ are in the row space of $\mathbf{A}$, which means that the row space of $\mathbf{B}$ is a subspace of $Row\mathbf{A}$.

Similarly the rows of $\mathbf{A}$ are in the row space of $\mathbf{B}$, which means that the row space of $\mathbf{A}$ is a subspace of $Row\mathbf{B}$.

This means that the row spaces are subspaces of each other, and are therefore equal.

$Row\,\underline{A} \subseteq Row\,\underline{B}$

$Row\,\underline{B} \subseteq Row\,\underline{A}$

This leads us to the following important statement:

> The non-zero row vectors of a matrix in echelon form comprise a basis for the row space of that matrix.

Example 1.4
Find a basis, and hence dimension, of the row space of the following matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 9 & 9 & 8 \\ 0 & -5 & -5 & -8 \\ 0 & -6 & -6 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Basis of $Row\,\underline{A}$ is,

$\{(1,3,1,3),(0,1,1,0),(0,0,0,1)\}$

$\Rightarrow \dim(Row\,\underline{A}) = 3$

Why column space changes!

$$x + y = 1$$

$$2x - 8y = 3 \longrightarrow x - 4y = \frac{3}{2}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -8 & 3 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -4 & \frac{3}{2} \end{array}\right)$$

Using the technique from the previous example we now also have another way of finding a basis for a spanning set.

Example 1.5
Find a basis, and hence dimension, of the subspace spanned by

$$S = \{(-1,2,5),(3,0,3),(5,1,8)\}$$

To do this we form rows from the vectors then reduce to echelon form:

$$\begin{pmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -5 \\ 0 & 6 & 18 \\ 0 & 11 & 33 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ a basis is $\{(1,-2,-5),(0,1,3)\}$

We can also use the same technique to find the basis of a column space simply by utilising the transpose.

Example 1.6
Find a basis, and hence dimension, of the column space of the matrix in Example 1.4.

$$\underline{A}^T = \begin{pmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 8 & -8 & -8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \text{Row}(\underline{A}^T) = \{(1,0,-3,3,2),(0,1,9,-5,-6),(0,0,1,-1,1)\}$$

$$\Rightarrow \text{Col}(\underline{A}) = \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim(\text{Col}\,\underline{A}) = 3$$

What we have just seen is always true - that the dimension of the row space and column space of a matrix are the same.

One final point before we move on is that, whilst doing row operations on a matrix changes the column space, it does not change the dependencies between columns of the original matrix.

> That means that from the row reduced matrix we can identify the linearly independent columns, and the corresponding columns in the original matrix form a basis for the column space.

We had,

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also LI ← Linearly independent (Leading ones)

which means the first, second and fourth columns of **A** are linearly independent and therefore constitute a basis for $Col\mathbf{A}$.

$$Col\ \underline{\underline{A}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

basis for column space

Let us now verify that what we have just written above is in fact a basis for $Col\mathbf{A}$.

Show $\underline{c}_3$ is linear combination of $\underline{c}_1, \underline{c}_2, \underline{c}_4$:

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & -1 & 6 \\ 3 & 4 & 1 & -2 \\ 2 & 0 & -2 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 9 & 8 & 9 \\ 0 & -5 & -8 & -5 \\ 0 & -6 & -8 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow x_3 = 0, \quad x_2 = 1, \quad x_1 = 1 - 3x_2 = -2$$

Check: $-2\begin{pmatrix}1\\0\\-3\\3\\2\end{pmatrix} + \begin{pmatrix}3\\1\\0\\4\\6\end{pmatrix} + \underline{0} = \begin{pmatrix}1\\1\\6\\-2\\-4\end{pmatrix}$ ✓

Since the third column is a linear combination of the other columns we can remove it and the columns still span $Col\mathbf{A}$.

Equally from the row reduction we have just applied the 3 columns left are linearly independent since the corresponding homogeneous system has only the trivial solution.

Therefore those columns do form a basis for the column space of $\mathbf{A}$.

1.3 Rank & Nullity

We define the ***rank*** of a matrix to be the dimension of the row (or column) space.

The ***nullity*** is defined to be the dimension of the null space of $\mathbf{A}$.

You can think of the null space as just the solution space to the homogeneous matrix equation. $\underline{A}\underline{x} = \underline{0}$

Example 1.7
Find the rank and nullity of,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\therefore \text{Rank } \underline{A} = 2}$$

$$x_3 = -x_4$$

$$x_1 = -2x_2 - 3x_4$$

(Homogeneous solution)

$$\therefore \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

basis

"basis for null space"

$\boxed{\therefore \text{Nullity} = 2}$

for solution space to homogeneous

The last example demonstrates the following important result.

> The number of columns of a matrix is equal to the rank + the nullity.

The next example will demonstrate how to use the facts that we have seen so far.

Example 1.8

Find a basis for the column space of the following matrix.

$\underline{\underline{A}}\underline{x} = \underline{b}$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Rank } \underline{\underline{A}} = 3$$

$\therefore \text{Nullity} = 5 - 3 = 2$

Linearly independent columns are $1^{st}, 2^{nd}, 4^{th}$.

Spanning set of Col $\underline{\underline{A}}$ is,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

So by just looking at the system matrix, we can now tell exactly what vectors, **b**, in the matrix equation, $\mathbf{A}\mathbf{x} = \mathbf{b}$, will have a solution, namely, anything in the column space which is any linear combination of the basis vectors we just found.

For example we can construct a vector from the column space basis which we know will have a solution for this particular system:

$$\underline{b} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \\ 3 \end{pmatrix}$$

$\Longrightarrow$ Can find solution to thi

$\Longrightarrow$ Will be a plane (Nullity

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & -1 & -3 & 1 & 3 & -1 \\ -2 & -1 & 1 & -1 & 3 & -3 \\ 0 & 3 & 9 & 0 & -12 & 3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 3 & -1 & -3 & 1 \\ 0 & -1 & -3 & 1 & 3 & -1 \\ 0 & 1 & 3 & 0 & -4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 3 & -1 & -3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & 3 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Longrightarrow x_3, x_5$ free

$$x_4 = x_5, \quad x_2 = 1 - 3x_3 + 4x_5$$

$$x_1 = 1 + 2x_3 - x_5$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \underbrace{x_3 \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \\ 1 \end{pmatrix}}$$

Particular Solution (Shift)

Spanning set for homogeneous part (Plane)

1.4 Coordinates & Change of Basis

We have already seen vectors and vectors spaces as geometric entities in the lower dimensions.

We will now formalise this and extend the ideas to higher dimensions.

> ***Definition***
>
> Let $B = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$ be an ordered basis for a vector space, V, and let $\mathbf{x}$ be a vector in V such that,
>
> $$\underline{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_p \underline{v}_p$$
>
> The scalars, c_i, with $i \in [1, p]$, are called the $\boxed{\textit{coordinates}}$ of $\mathbf{x}$ relative to the basis, B.
>
> The column vector containing these coordinates is known as the ***coordinate matrix/vector*** of $\mathbf{x}$ relative to B:
>
> $$[\underline{x}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}$$

Example 1.9
Find the coordinate matrix of $\mathbf{x} = (-2, 1, 3)$ *in* $\mathbb{R}^3$ *relative to the standard basis.*

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$\underline{x} = -2(1,0,0) + 1(0,1,0) + 3(0,0,1)$$

$$\therefore [\underline{x}]_S = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

In other words the coordinates relative to the standard basis are just the components of $\mathbf{x}$.

Example 1.10
Let $B = \{\mathbf{v_1}, \mathbf{v_2}\} = \{(1,0), (1,2)\}$ *be a basis for* $\mathbb{R}^2$. *If the coordinates of a point relative to this basis are given by,*

$$[\mathbf{x}]_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

then find the coordinates relative to the standard basis.

$$\underline{x} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$S = \{(1,0), (0,1)\}$$

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} \therefore [x]_S = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

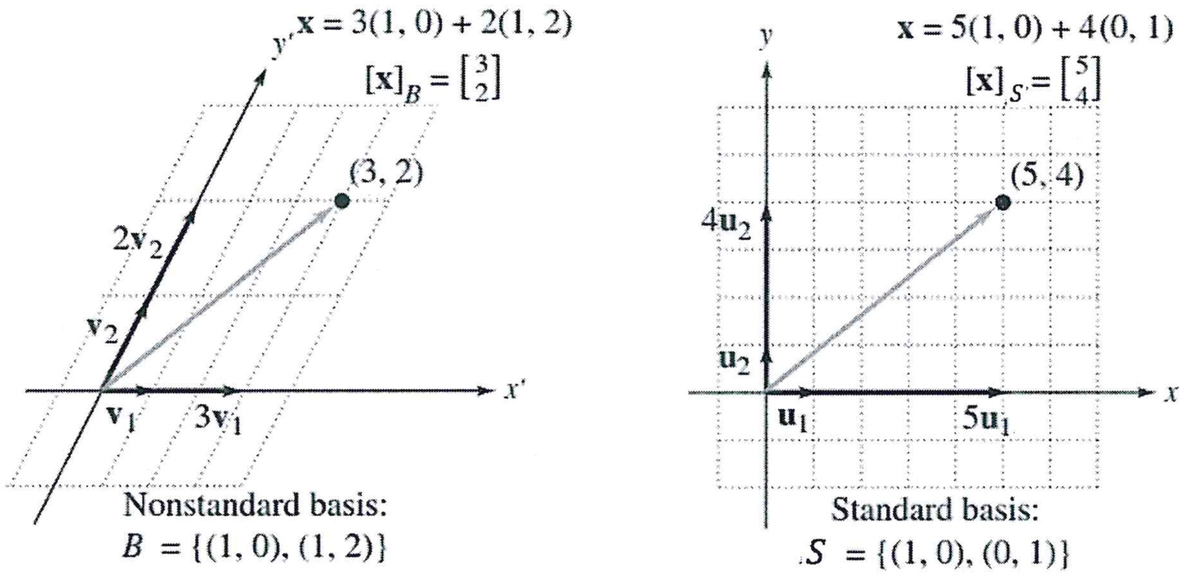

Figure 1: Figure for Example 1.10 ©Houghton Mifflin Harcourt Publishing Company, 2009.

Example 1.11
Find the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ *relative to the basis* $B = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \{(1,0,1), (0,-1,2), (2,3,-5)\}$.

$$\underset{\underline{\underline{P}}}{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{pmatrix}} \underset{[x]_B}{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}} = [x]_S$$

$$(1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 2 & -7 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right) \Longrightarrow [x]_B = \begin{pmatrix} 5 \\ -8 \\ -2 \end{pmatrix}$$

The procedure from the last two examples is called a ***change of basis***.

In other words given the coordinates of a vector in one basis, we find its coordinates relative to another.

Breaking down what we did, we had define the ***transition matrix*** from a basis, B, to another basis, B', to be $\mathbf{P}$, such that:

$$\underline{\underline{P}}[x]_B = [x]_{B'} \iff [x]_B = \underline{\underline{P}}^{-1}[x]_{B'}$$

So instead of doing a row reduction every time we want to write a new vector from the standard basis in terms of another basis we can just find the inverse transition matrix and do matrix multiplication to obtain the coordinates.

Let us do this for the previous example:

$$\underline{\underline{P}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{pmatrix} \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 2 & -7 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right)$$

$$\therefore \underline{\underline{P}}^{-1} = \begin{pmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{pmatrix} \Longrightarrow [x]_B = \underline{\underline{P}}^{-1}[x]_{B'}$$

standard basis, S

$$= \begin{pmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \\ -2 \end{pmatrix}$$

Now we can write any vector relative to that basis by simple matrix multiplication instead of row reduction.

We will omit the proof that $\mathbf{P}$ is invertible, however the main idea is this:

- A basis is composed of linearly independent vectors.
- The matrix with columns given by the basis vectors has rank, n.
- Row operations preserve the rank.
- The only $n \times n$ matrix of rank, n, is the identity matrix.
- If the reduced echelon form of a matrix is the identity matrix then it is invertible.

In the previous example we took a vector in the standard basis and found the transition matrix for another basis.

We can extend this technique to find the transition matrix between 2 non-standard bases.

Say we 2 bases, B' and B. Relative to the standard basis we have,

$$\underline{P}\,[\underline{x}]_B = [\underline{x}]_S = \underline{Q}\,[\underline{x}]_{B'}$$

$$\underline{E}_k \cdots \underline{E}_2 \underline{E}_1 \underline{P} = \underline{I}$$

$$\therefore \underline{P}^{-1} = \underline{E}_k \cdots \underline{E}_2 \underline{E}_1$$

where $\mathbf{P}$ is the transition matrix from B to S, and $\mathbf{Q}$ is the transition matrix from B' to S.

Remember, $\mathbf{P}$, is composed of the vectors in the basis, B, and $\mathbf{Q}$ is composed of the vectors in the basis, Q.

Equating the leftmost and rightmost parts of the above equation we have,

$$\underline{P}\,[\underline{x}]_B = \underline{Q}\,[\underline{x}]_{B'}$$

$$[\underline{x}]_B = \underline{P}^{-1}\underline{Q}\,[\underline{x}]_{B'}$$

$$= \underline{E}_k \cdots \underline{E}_2 \underline{E}_1 \underline{Q}\,[\underline{x}]_{B'}$$

row reduce $\underline{Q}$

where the elementary matrices above correspond with the row operations that reduce $\mathbf{P}$ to the identity matrix.

In other words, if we apply those same elementary row operations to the matrix, $\mathbf{Q}$, then we obtain the transition matrix from the basis, B', to the basis, B.

The transition matrix, **R**, from B' to B can be found using Gauss-Jordan elimination as follows:

B': Q $\quad$ B: P

$$(\underline{\underline{P}} \mid \underline{\underline{Q}}) \to (\underline{\underline{I}} \mid \underline{\underline{R}})$$

Example 1.12

Find the transition matrix from B to B' for the following bases in $\mathbb{R}^2$.

$$B = \{(-3,2),(4,-2)\}$$
$$B' = \{(-1,2),(2,-2)\}$$

Have B
need B'

$B \to S$: $\underline{\underline{P}} = \begin{pmatrix} -3 & 4 \\ 2 & -2 \end{pmatrix}$

$B' \to S$: $Q = \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}$

$B \to B'$: $\underline{\underline{P}}[\underline{x}]_B = \underline{\underline{Q}}[\underline{x}]_{B'} \Rightarrow$ Need $[\underline{x}]_{B'}$

$$[\underline{x}]_{B'} = \underline{\underline{Q}}^{-1}\underline{\underline{P}}[\underline{x}]_B$$

$$\left(\begin{array}{cc|cc} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -2 & 3 & -4 \\ 0 & 2 & -4 & 6 \end{array}\right)$$

$\underline{\underline{Q}}$ $\quad$ $\underline{\underline{P}}$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right)$$

$\therefore$ transition matrix: $\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$

$$[\underline{x}]_{B'} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}[\underline{x}]_B$$

$$1(1,0,0)+2(0,1,0)-1(0,0,1)$$
$$= c_1(1,0,1)+c_2(0,-1,2)$$
$$+c_3(2,3,-5)$$