Vector Spaces (Part I)

Conceptual knowledge

- Revise vector spaces, subspaces, span and linear independence.
- Understand null spaces, row/column spaces and bases.

Procedural knowledge

- Determine whether a set of vectors forms a vector space/subspace or not.
- Find spanning sets.

Practice Problems

- Determine whether a vector is part of a given null/row/column space.
- Determine linear independence/dependence of vectors and writing vectors as a linear combination.
- Find bases of vector spaces. (Next week)

1 Vector Spaces

Vectors are encountered frequently in science and engineering.

Quite often a set of vectors which share certain properties arise, allowing us to do useful mathematics when analysing systems.

The fundamental type of sets of vectors that we will now explore is vector spaces.

You have already seen vector spaces in geometry and Fourier analysis. We now formalise these underlying structures and subject them to analysis in their own right.

Definition

A ***vector space*** is a nonempty set, V, of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

rule

Compose Boolean Algebra

1) ***Closed*** under addition: $(\mathbf{u}+\mathbf{v}) \in V$.
2) ***Commutative*** under addition: $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3) ***Associative*** under addition: $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4) Additional identity: $\mathbf{0} \in V$, such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ "Additive identity"
5) Additional inverse: $\forall \mathbf{u} \in V\, \exists(-\mathbf{u})$, such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ "Additive inverse"
6) ***Closed*** under scalar multiplication: $c\mathbf{u} \in V$.
7) ***Distributive*** under vector addition: $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
8) ***Distributive*** under scalar addition: $(c+d)\mathbf{u}=\cancel{c\mathbf{u}+c\mathbf{v}}$ $c\underline{u}+d\underline{u}$
9) ***Associative*** under scalar multiplication: $c(d\mathbf{u})=(cd)\mathbf{u}$
10) Multiplicative identity: $1\mathbf{u}=\mathbf{u}$

Example 1.1

objects

The set of vectors, $\mathbf{v} \in \mathbb{R}^n$, where n is a finite, positive number, along with vector addition and scalar multiplication is a vector space.

Operations

We can verify this by going through the list of axioms and using the definition of a vector and the properties of real numbers.

For example let us look at axioms 1 and 4.

1) Closed under addition: Let, $\underline{u}=(u_1,u_2,\ldots,u_n)$
$\underline{v}=(v_1,v_2,\ldots,v_n)$

$$\underline{u}+\underline{v}=(u_1,u_2,\ldots,u_n)+(v_1,v_2,\ldots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\ldots,u_n+v_n)$$
$$(u_i+v_i)\in\mathbb{R} \quad \forall i\in[1,n]$$
$$\Longrightarrow (\underline{u}+\underline{v})\in\mathbb{R}^n$$

4) With $\underline{0}=(0,0,\ldots,0)$, we have,

$$\underline{u}+\underline{0}=(u_1+0,u_2+0,\ldots,u_n+0)=\underline{u}$$

etc.

Example 1.2 objects Operations

The set of all 2×3 matrices, M_{23}, with matrix addition and scalar multiplication is a vector space.

Axioms 4) & 6)

Let, $\underline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, $\underline{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

4) With $\underline{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we have,

$$\underline{A} + \underline{0} = \begin{pmatrix} a_{11}+0 & a_{12}+0 & a_{13}+0 \\ a_{21}+0 & a_{22}+0 & a_{23}+0 \end{pmatrix} = \underline{A} \checkmark$$

6) Let $c \in \mathbb{R}$,

$$c\underline{A} = c\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}$$

$$ca_{ij} \in \mathbb{R} \quad \forall i \in [1,2],\ j \in [1,3]$$

$$\therefore c\underline{A} \in M_{23} \checkmark \qquad \underline{\text{etc.}}$$

Similarly all $m \times n$ rectangular and $n \times n$ square matrices are vector spaces.

Example 1.3

The set of all polynomials to the n^{th} degree or less, P_n, with addition and multiplication defined as follows:

$$p(x) = a_0 + a_1 x + \dots + a_n x^n, \quad q(x) = b_0 + b_1 x + \dots + b_n x^n$$

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$$

$$c\,p(x) = ca_0 + ca_1 x + \dots + ca_n x^n$$

The axioms can be shown to hold for these polynomials using the properties of real numbers.

For example consider axiom 7:

$$c(p(x)+q(x)) = c\left[(a_0+b_0)+(a_1+b_1)x+\dots+(a_n+b_n)x^n\right]$$
$$= (ca_0+ca_1x+\dots+ca_nx^n)+(cb_0+cb_1x+\dots+cb_nx^n)$$
$$= c.p(x)+cq(x)$$

etc.

Example 1.4
The set of all real-valued continuous functions over the real line, $C(-\infty,\infty)$, with addition and multiplication defined as follows:

Objects

$$(f+g)(x) = f(x)+g(x)$$
$$(cf)(x) = c\left[f(x)\right]$$

Operations

Closure under addition and multiplication is easy to show using properties of continuous functions on the real line (from basic calculus).

In order to satisfy axiom 4 we must define the additional identity as,

$$f_0(x) = 0 \quad \forall x$$
$$\Longrightarrow (f+f_0)(x) = f(x)+f_0(x) = f(x)+0 = f(x)$$

The other axioms follow from the properties of continuous functions.

The above examples are commonly occurring vector spaces. Let's now look at some examples of things which are not vector spaces.

Example 1.5
The set of integers, $\mathbb{Z}$, with standard addition and multiplication is not a vector space.

6) $c\underline{u} \in V$? : $c=\frac{1}{2} \in \mathbb{R}$, $\underline{u}=1 \in \mathbb{Z}$

$$c\underline{u} = \tfrac{1}{2} \notin \mathbb{Z}$$

(Counter example)

$\therefore$ Not closed under scalar multiplication

$\therefore$ not a vector space.

Example 1.6
The set of polynomials of exactly degree 2 with operations defined similarly to Example 1.3.

P_2^*: 1) Closed under addition: $p(x) = x^2$
$q(x) = -x^2 + x + 1$
$\Longrightarrow p(x) + q(x) = x + 1 \notin P_2^*$ (degree 1)
$\therefore$ Not closed under addition, $\therefore$ not a vector space.

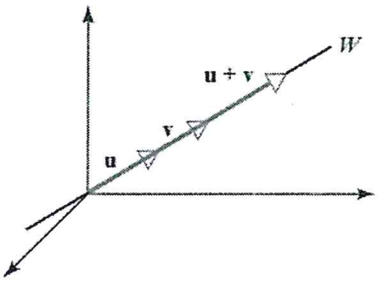
1.1 Subspaces

> ***Definition***
>
> A ***subspace*** of a vector space, V, is a subset, H, of V that has the following 3 properties:
>
> a) The zero vector of V is in H.
>
> b) $\mathbf{u} + \mathbf{v}$ is in H. Closed under addition and scalar multiplication
>
> c) $c\mathbf{u}$ is in H for all scalars, c, and vectors $\mathbf{u}$ in H.

In other words the subset must be closed under addition and scalar multiplication, containing the zero vector.

In fact if we can show b) and c) is true then a) is automatically satisfied since we can choose $c = 0$ so that $c\mathbf{u} = \mathbf{0}$.

Example 1.7
Lines through the origin are subspaces of $\mathbb{R}^2$ or $\mathbb{R}^3$. Compare with Q7 from practice problems



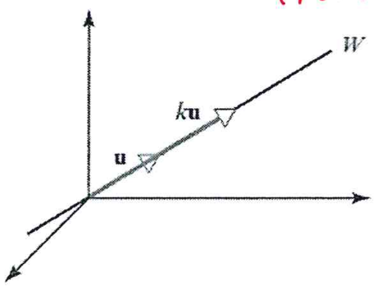


Figure 1: Lines through the origin closed under addition and scalar multiplication © Wiley, 2013.

Example 1.8
Let W be the set of symmetric 2 × 2 matrices with the standard operations. Is W a subspace?

$\Rightarrow$ M_{22} is a vector space.

$\Rightarrow$ Symmetric 2x2 matrices are a subset of M_{22}.

$$W \subseteq M_{22}$$

1) Closed under addition:

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_3+b_3 \\ a_2+b_2 & a_4+b_4 \end{pmatrix}$$

$$(\underline{\underline{A}} + \underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T = (\underline{\underline{A}} + \underline{\underline{B}}) \in W \checkmark$$

2) Closed under scalar multiplication:

$$(c\underline{\underline{A}})^T = c(\underline{\underline{A}}^T) = c(\underline{\underline{A}}) = (c\underline{\underline{A}}) \in W \checkmark$$

$\therefore$ W is a subspace.

Example 1.9
Let W be the set of singular 2 × 2 matrices with the standard operations. Is W a subspace?

det = 0

1) Let, $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\underline{\underline{B}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ Both singular

$\underline{\underline{A}} + \underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $|\underline{\underline{A}} + \underline{\underline{B}}| \neq 0$ $\therefore$ invertible,

$\therefore$ Not closed under addition

$\therefore$ Not a subspace.

Example 1.10

Which of the following subsets is a subspace of $\mathbb{R}^2$*?*

1. $x + 2y = 0$

2. $x + 2y = 1$

(x_1, y_1) (x_2, y_2)

1) Addition: $x_1 + 2y_1 = 0, \quad x_2 + 2y_2 = 0$

$$(x_1 + x_2, y_1 + y_2) \Rightarrow (x_1 + x_2) + 2(y_1 + y_2)$$
$$= (x_1 + 2y_1) + (x_2 + 2y_2) = 0$$

Multiplication: $(cx_1, cy_1) \Rightarrow cx_1 + 2cy_1$

$$= c(x_1 + 2y_1) = c(0) = 0$$

$\therefore$ It is a subspace.

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2) $(x_1 + x_2, y_1 + y_2) \Rightarrow (x_1 + x_2) + 2(y_1 + y_2)$

$$= (x_1 + 2y_1) + (x_2 + 2y_2) = 1 + 1 = 2$$

Not closed under addition

$\therefore$ not a subspace.

Alternatively, for the second case we could have also just observed that the zero vector is not in the subset and therefore cannot be a subspace.

1.2 Linear Independence & Spanning Sets

> ***Definition***
>
> A set of vectors, $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$, in $\mathbb{R}^n$ is ***linearly independent*** if,

$$x_1 \underline{v}_1 + x_2 \underline{v}_2 + \dots + x_p \underline{v}_p = \underline{0}$$

has only the trivial solution. (linearly independent)

It is ***linearly dependent*** if there exist values of $x_1 = c_1$, $x_2 = c_2$, ..., $x_p = c_p$, not all zero, such that the above equation holds.

The values c_1, c_2, ..., c_p are the weights of each vector and the sum is known as a ***linear combination***.

It is common to say that the vectors, $\mathbf{v_1}, \dots, \mathbf{v_p}$, are linearly independent when the set of those vectors is linearly independent.

Example 1.11
Is the following set of vectors linearly independent?

$$S = \{\underset{\underline{v}_1}{(1,3,1)}, \underset{\underline{v}_2}{(0,1,2)}, \underset{\underline{v}_3}{(1.0,-5)}\}$$

$\underline{v}_1$ is linear combination of $\underline{v}_2$ and $\underline{v}_3$:

$$\underline{v}_1 = 3\underline{v}_2 + \underline{v}_3 = 3(0,1,2) + (1,0,-5)$$
$$= (1,3,1) = \underline{v}_1$$

$\therefore$ Linearly dependent.

Sometimes, as in the last example, we can see from inspection whether the vectors can be written as a linear combination of the others or not.

A more robust method is to use Gauss-Jordan elimination.

Example 1.12
Is the following set of vectors linearly independent?

$$S = \{(1,2,3), (0,1,2), (-1,0,1)\}$$

For linear independence we require,

Require, $c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}0\\1\\2\end{pmatrix} + c_3\begin{pmatrix}-1\\0\\1\end{pmatrix} = \underline{0}$

only has trivial solution.

This can be written as a system of linear equations and solved as follows:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \infty \text{ solutions}$$

$\therefore$ linearly dependent.

Example 1.13

Can the vector $(1, 1, 1)$ *be written as a linear combination of the vectors in* S *from the previous example? If so, find the coefficients.*

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}0\\1\\2\end{pmatrix} + c_3\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \infty \text{ solutions}$$

c_3 is free variable.

$$c_2 = -1 - 2c_3$$

$$c_1 = 1 + c_3$$

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1+c_3\\-1-2c_3\\c_3\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

Check! For example if $c_3 = 1 \Longrightarrow c_1 = 2, c_2 = -3$

$$2\begin{pmatrix}1\\2\\3\end{pmatrix} - 3\begin{pmatrix}0\\1\\2\end{pmatrix} + \begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

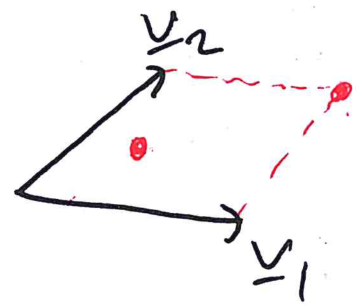
$$c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} + c_3\begin{pmatrix}1\\-1\end{pmatrix} = \underline{0}$$

No. of vectors : 3
No. of entries : 2

$$c_1 + c_3 = 0$$
$$c_2 - c_3 = 0$$

$\Rightarrow c_3$ is free, ∞ solutions

$\therefore$ linearly dependent.



$$S = \{\underline{v}_1, \underline{v}_2\}$$

$$\operatorname{Span} S = c_1\underline{v}_1 + c_2\underline{v}_2$$

Note that any set of vectors which has more vectors than entries in each vector is linearly dependent since the resulting homogeneous system will always have a free variable.

Definition

Given a set of vectors, $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$, in $\mathbb{R}^n$, the ***span*** of the set is the collection of all linear combinations of those vectors denoted by,

$$\text{Span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_p v_p$$

where the c_i, $i \in [1, p]$, are scalars.

Example 1.14

Is $(-3, 8, 1)$ *in Span*$\{(1, -2, 3), (5, -13, -3)\}$*?*

$$c_1\begin{pmatrix}1\\-2\\3\end{pmatrix} + c_2\begin{pmatrix}5\\-13\\-3\end{pmatrix} = \begin{pmatrix}-3\\8\\1\end{pmatrix} \ ?$$

$$\left(\begin{array}{cc|c}1 & 5 & -3\\-2 & -13 & 8\\3 & -3 & 1\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 5 & -3\\0 & -3 & 2\\0 & -18 & 10\end{array}\right)$$

$$\sim \left(\begin{array}{cc|c}1 & 5 & -3\\0 & 1 & -2/3\\0 & 0 & -2\end{array}\right) \Longrightarrow \text{Inconsistent} \therefore \text{no solution}$$

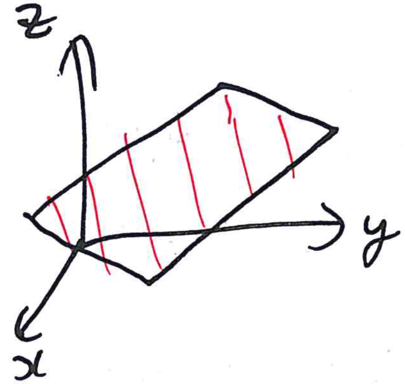
$$\therefore \begin{pmatrix}-3\\8\\1\end{pmatrix} \text{ is not in the span.}$$

Example 1.15

Determine whether the following sets span $\mathbb{R}^3$ *or not.*

1. $S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$

2. $S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$



$ax + by + cz = k \ (\in \mathbb{R})$

1) $c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}0\\1\\2\end{pmatrix} + c_3\begin{pmatrix}-2\\0\\1\end{pmatrix} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$, $v_1, v_2, v_3 \in \mathbb{R}$,

$$\left(\begin{array}{ccc|c}1 & 0 & -2 & v_1\\2 & 1 & 0 & v_2\\3 & 2 & 1 & v_3\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & -2 & v_1\\0 & 1 & 4 & v_2-2v_1\\0 & 2 & 7 & v_3-3v_1\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c}1 & 0 & -2 & v_1\\0 & 1 & 4 & v_2-2v_1\\0 & 0 & -1 & v_3-2v_2+v_1\end{array}\right) \Longrightarrow \text{Unique solution}$$

$\therefore S_1$ spans $\mathbb{R}^3$.

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2) $$\left(\begin{array}{ccc|c}1 & 0 & -1 & v_1\\2 & 1 & 0 & v_2\\3 & 2 & 1 & v_3\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & -1 & v_1\\0 & 1 & 2 & v_2-2v_1\\0 & 2 & 4 & v_3-3v_1\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c}1 & 0 & -1 & v_1\\0 & 1 & 2 & v_2-2v_1\\0 & 0 & 0 & v_3-2v_2+v_1\end{array}\right) \Longrightarrow \text{Only solution when } v_3-2v_2+v_1=0$$

$\therefore$ does not span $\mathbb{R}^3$.

Spans a plane in $\mathbb{R}^3$.

Note that all spanning sets are subspaces by definition.

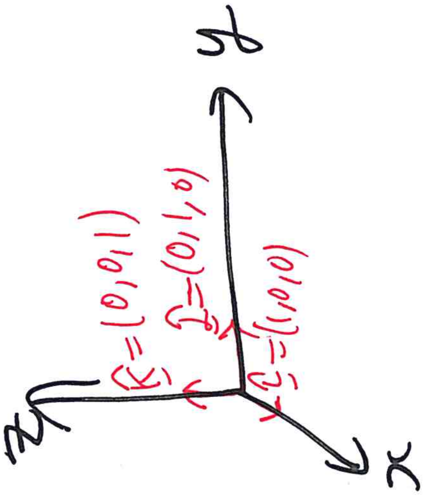
Definition

A set of vectors, $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_p}\}$, in a vector space, V, is called a ***basis*** for V if,

1. S spans V.
2. S is linearly independent.

The plural of basis is ***bases*** (say "bay-sees").

Example 1.16
Show that $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ *is a basis for* $\mathbb{R}^3$.


y
z
x
k = (0,0,1)
j = (0,1,0)
i = (1,0,0)

Check it spans $\mathbb{R}^3$ and linearly independent.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & v_1 \\ 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & v_3 \end{array}\right) \Longrightarrow \text{Spans } \mathbb{R}^3 \quad \because \text{ unique solution.}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Longrightarrow \text{No solution except trivial solution}$$

$\therefore$ Linearly independent.

$\therefore$ S is a basis for $\mathbb{R}^3$.

This particular basis is known as the ***standard basis*** for $\mathbb{R}^3$.

Analagous standard basis can be defined similarly for other dimensions.

Example 1.17
Show that $S = \{(1, 1), (1, -1)\}$ is a non-standard basis for $\mathbb{R}^2$. (Standard: $\{(1,0),(0,1)$

$$\left(\begin{array}{cc|c} 1 & 1 & v_1 \\ 1 & -1 & v_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & v_1 \\ 0 & -2 & v_2 - v_1 \end{array}\right) \Longrightarrow \text{Unique solution}$$

$\therefore$ Spans $\mathbb{R}^2$.

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right) \Longrightarrow \text{Only trivial solution}$$

$\therefore$ Linearly independent.

$\therefore$ S is a basis for $\mathbb{R}^2$.

Example 1.18
Show that the set of polynomials of degree 3 has a basis,

$$S = \{1, x, x^2, x^3\}$$

Span $S = c_1 + c_2 x + c_3 x^2 + c_4 x^3$
which is also the definition of 3^{rd} degree polynomial.

S spans P_3^*.

But linear independence requires,

$$c_1 + c_2 x + c_3 x^2 + c_4 x^3 = 0$$

True only if $c_1 = c_2 = c_3 = c_4 = 0 \quad \forall x$.

Only trivial solution $\therefore$ Linearly independent.

$\therefore S$ is a basis for P_3^*.

Theorem (Uniqueness of Basis Representation)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a basis for the vector space V then every vector in V can be written in one, and only one, way as a linear combination of vectors in S.

Proof

Existence: S is a basis $\Rightarrow$ vectors in S span V.
So every vector in V is a linear combination of vectors in S.

Uniqueness: $\underline{u} \in V$ can be represented as,

$$\underline{u} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \ldots + c_p \underline{v}_p$$

Suppose there is another representation! (Null hypothesis)

$$\underline{u} = b_1 \underline{v}_1 + b_2 \underline{v}_2 + \ldots + b_p \underline{v}_p$$

then, $\underline{u} - \underline{u} = (b_1 - c_1)\underline{v}_1 + (b_2 - c_2)\underline{v}_2 + \ldots + (b_p - c_p)\underline{v}_p = \underline{0}$

true for all vectors when $b_i = c_i \quad \forall i \in [1, p]$.

$\therefore$ Unique representation. $\square$

It is now easy enough to see the following:

> If a basis for a vector space, V, contains p vectors, then any subset of V with more than p vectors is linearly dependent.
>
> Similarly, any other basis that can be found will also have p vectors.

1.3 Row, Column and Null Spaces

> ***Definition***
>
> Given a matrix, $\mathbf{A}$, the ***row space*** is defined to be the subspace spanned by the rows of $\mathbf{A}$.
>
> The ***column space*** is defined to be the subspace spanned by the columns of $\mathbf{A}$.
>
> The ***null space*** is defined to be the subspace corresponding with the solutions to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$.

Since we can write a matrix equation as,

$$\underline{A}\,\underline{x} = \underline{b}$$

$$x_1 \underline{c}_1 + x_2 \underline{c}_2 + \dots + x_n \underline{c}_n = \underline{b}$$

where the $\mathbf{c_i}$ are the columns of $\mathbf{A}$, we conclude that the system is consistent if and only if $\mathbf{b}$ is in the column space of $\mathbf{A}$.

Furthermore, the solution to the system can be written as the sum of a ***particular solution*** to $\mathbf{Ax} = \mathbf{b}$ and the ***general solution*** to $\mathbf{Ax} = \mathbf{0}$.

In particular, if $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ is a basis for the null space of $\mathbf{A}$, then the full solution to the nonhomogeneous matrix equation is,

$$\underline{x} = \underbrace{\underline{x}_0}_{\text{Particular solution}} + \underbrace{(a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_k \underline{v}_k)}_{\text{homogeneous part}}$$

where $\mathbf{x_0}$ is any solution to the equation.

Note that row operations do not affect the row or null space, however they do affect the column space. More on this topic next week.