The way to think about it is:

- Elementary matrices are only one step away from the identity matrix.
- If we want to find the inverse of the elementary matrix we need to find the step to apply which brings us back to the identity matrix.
- The elementary matrix for this return step is the inverse of the original elementary matrix.

---

e.g.

$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Let's do a row operation on $\mathbf{A}$, say $R_2 \to R_2 - 3R_1$:

$\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$

The corresponding elementary matrix for this operation is,

$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$

If we want the inverse, $\mathbf{E}^{-1}$, then we should think about how to return $\mathbf{E}$ the identity matrix, $\mathbf{I}$.

$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The row operation required is $R_2 \to R_2 + 3R_1$ and is the inverse elementary matrix we are looking for:

$\mathbf{E}^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

You can check this as follows:

$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \mathbf{A}$

Let's do an example with multiplication:

e.g.

$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Let's do a row operation on $\mathbf{A}$, say $R_1 \to 2R_1$:

$\begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}$

The corresponding elementary matrix for this operation is,

$\mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

If we want the inverse, $\mathbf{E^{-1}}$, then we should think about how to return $\mathbf{E}$ the identity matrix, $\mathbf{I}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The row operation required is $R_2 \rightarrow R_2/2$ and is the inverse elementary matrix we are looking for:

$$\mathbf{E^{-1}} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

You can check this as follows:

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \mathbf{A}$$