Linear Systems and Matrices

Conceptual knowledge

- Revise linear systems & Gauss-Jordan elimination.
- Understand elementary matrices and LU factorisation.

Procedural knowledge

- Construct a matrix equation from a system of linear equations.
- Reduce a rectangular matrix to echelon and reduced echelon form.
- Find the inverse of a matrix by Gauss-Jordan elimination.
- Write a matrix as a product of elementary matrices or LU factors.
- Solve linear systems using LU factorisation.

1 Linear Systems

Recall that a ***linear equation*** is of the form,

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where b and each a_i, $i \in [1, n]$, are real or complex numbers known as coefficients. We will focus on only systems with real coefficients.

So linear equations cannot have any variable raised to a power not equal to 1, can not be multiplied by any other variable to any power except 0, and can not be used in a transcendental function.

We define a ***linear system*** to be a collection of linear equations.

A ***solution*** to a linear system is a sequence of numbers, s_1, s_2, ..., s_n, that when substituted into the system satisfies each equation.

The set of all such sequences is called the ***solution set*** of the system.

A system which has an empty solution set we say has ***no solution***, and we call the system of equations ***inconsistent***.

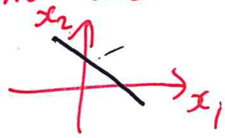
If the solution set has only one sequence of numbers which satisfy the system then it is a ***consistent system*** with a ***unique solution***.

Finally if the solution set has more than one sequence of numbers which satisfy the system then it must have an infinite number of them and we say it has ***infinitely many solutions***.

We can also represent these infinite solutions ***parametrically***.

Example 1.1
Find the solution to the following system.

$$\left.\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + 2x_2 &= 6 \end{aligned}\right\} \text{ same straight line}$$



$x_1 = 3 - x_2 \longrightarrow$ Satisfies both equations

$\therefore$ infinite solutions depending on x_2.

Parametrically! $x_1 = 3 - t$, $x_2 = t$, $t \in \mathbb{R}$

In what follows it will be useful to write systems or linear equations using the coefficient matrix and the augmented matrix.

Note that often when typsetting the augmented matrix the vertical line is omitted, however by hand it is usually clearer to include it.

Example 1.2

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 3 \\ -3x_2 + 7x_3 &= -1 \\ 2x_1 + 4x_2 - x_3 &= 0 \end{aligned}$$

$$\underset{=}{A} \qquad \underset{=}{b}$$

$$\begin{pmatrix} 1 & -2 & 4 \\ 0 & -3 & 7 \\ 2 & 4 & -1 \end{pmatrix}, \quad \left(\begin{array}{ccc|c} 1 & -2 & 4 & 3 \\ 0 & -3 & 7 & -1 \\ 2 & 4 & -1 & 0 \end{array}\right)$$

Coefficient matrix Augmented matrix

$\underset{=}{A}\underset{-}{x} = \underset{-}{b} \leftarrow$ matrix equation

1.1 Solving Systems of Linear Equations

We know it is possible to solve systems of linear equations ***using matrices***:

$$\underline{\underline{A}}\,\underline{x} = \underline{b} \implies \underline{\underline{A}}^{-1}\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$$

$$\therefore \underline{x} = \underline{\underline{A}}^{-1}\underline{b}$$

We also have ***Cramer's rule***, which is useful when it comes to solving nonhomogeneous differential equations (see variation of parameters).

However the most robust method is by ***Gaussian*** or ***Gauss-Jordan elimination*** (also known as row reduction) since it can determine not just unique solutions, but the method will also tell you whether a system is inconsistent or has infinitely many solutions.

It is also easy to extract the parameterisation from the final step of the algorithm.

Gaussian elimination is the name given to the process of using row reduction to reduce a matrix to ***echelon form*** in order to solve the system by back substitution.

Gaus-Jordan elimination is the same process but taken to the ***reduced echelon form*** in order to simply read off the solutions.

Let us now review echelon and reduced echelon forms.

Definition

A rectangular matrix is in ***echelon form*** (or row echelon form) if it has the following three properties:

1) All non-zero rows are above any rows of all zeros.

2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3) All entries in a column below a leading entry are zeros.

   If a matrix in echelon form satisfies the following additional conditions, then it is in ***reduced echelon form*** (or reduced row echelon form):

4) The leading entry in each non-zero row is 1.

5) Each leading 1 is the only non-zero entry in its column.

In the following examples $\square$ represents the leading entry of each row which is a non-zero number and $\times$ represents any non-zero number that is not a leading entry.

Example 1.3

The following matrices are in echelon form:

$$\begin{pmatrix} \square & \times & \times & \times \\ 0 & \square & \times & \times \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \square & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & \square & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 & \square & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & \square & \times \end{pmatrix}$$

Example 1.4

The following matrices are in reduced echelon form:

$$\begin{pmatrix} 1 & 0 & \times & \times \\ 0 & 1 & \times & \times \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & \times & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 1 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 1 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & 1 & \times \end{pmatrix}$$

Each matrix has a unique reduced echelon form.

We know that performing basic operations on rows preserves the magnitude of the determinant of a matrix.

Similarly these basic row operations preserve the relative quantities of a matrix which allows us to reduce it to echelon form.

The echelon form then represents the solution to the system of equations represented by the augmented matrix.

Before we do an example let us introduce another definition.

Definition

A ***pivot position*** in a matrix, $\mathbf{A}$, is a location that corresponds to a leading 1 in the reduced echelon form of $\mathbf{A}$. A ***pivot column*** is a column that contains a pivot position.

Let's now introduce the row reduction algorithm for finding the reduced echelon form of a matrix.

Row Reduction Algorithm

1) Start with the leftmost non-zero column. This is a pivot column with pivot position at the top.

2) Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

3) Use row operations to create 0's in all positions below the pivot.

4) Ignore the row containing the pivot and all rows above it, if any. Apply steps 1-3 to the submatrix that remains. Repeat this process until there are no more non-zero rows to modify.

5) To obtain the reduced echelon form make all entries above a pivot zero by using row operations (starting with the rightmost) then scale each row by dividing by the respective pivots in order to obtain 1 as the leading entry in each row.

Example 1.5
Find the reduced echelon form of the following matrix.

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

pivot

1), 2) $\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$

3) $R_2 \to R_2 - R_1$: $\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$

4) Ignore R_1, repeat 1)–3) for other rows.

$R_3 \to \frac{-3}{2}R_2 + R_3$: $\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$ ← An echelon form

Right most pivot

Get reduced form!

$R_1 \to R_1 - 6R_3$, $R_2 \to R_2 - 2R_3$: $\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

$R_2 \to \frac{R_2}{2}$: $\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

$R_1 \to R_1 + 9R_2$: $\begin{pmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

$R_1 \to \frac{R_1}{3}$: $\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

Reduced echelon form

It is now appropriate to introduce the symbol "$\sim$" to mean "row equivalent".

In these notes, sometimes calculations will be shortened by use of this symbol, however for practice you should compute the row reduction yourself.

Example 1.6

Solve the following system using Gaussian elimination and back substitution.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right)$$

An echelon form

$$\Longrightarrow \quad \begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ -13x_4 &= -39 \Rightarrow x_4 = 3 \end{aligned}$$

"back substitution"

$$x_4 = 3, \quad x_3 = 1, \quad x_2 = 2, \quad x_1 = -1$$

If you get a false statement in your echelon form then it tells you that the system is inconsistent.

Example 1.7

The following augmented matrix which represents a linear system has been reduced to echelon form. The form shows the system is inconsistent (no solution).

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right) \Longrightarrow 0\times x_1 + 0\times x_2 + 0\times x_3 = -2$$

False statement

A row of zeros indicates infinite solutions.

Example 1.8

The following augmented matrix which represents a linear system has been reduced to echelon form. The form shows the system has infinite solutions.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \begin{array}{c} x_1 - x_2 + 2x_3 = 4 \\ x_2 - x_3 = 2 \\ 0\times x_1 + 0\times x_2 + 0\times x_3 = 0 \end{array}$$

True $\forall x_3$

x_1, x_2 then depend on x_3

In the last example we say that x_3 is a ***free variable*** since we can choose it to be anything.

We say that x_1 and x_2 are ***basic variables*** since they are determined by the free variables.

It is convention that we label the basic variables as the ones in the pivot positions.

We will now look at an example where we solve using Gauss-Jordan elimination.

This form will allow us to easily read off the solutions to the system as well as write the solution in vector form when there are infinite solutions.

Example 1.9
Use Gauss-Jordan elimination to show that the following system has a unique solution.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 9 \\ -x_1 + 3x_2 &= -4 \\ 2x_1 - 5x_2 + 5x_3 &= 17 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$ Echelon form

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$ Reduced echelon form

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 2$$

Definition

A ***homogeneous*** linear system of equations is of the form,

$$\underline{\underline{A}}\,\underline{x} = \underline{0} \leftarrow (0, 0, \ldots, 0)$$

A ***non-homogeneous*** linear system of equations is of the form,

$$\underline{\underline{A}}\,\underline{x} = \underline{b}, \quad \underline{b} \neq \underline{0}$$

Note that homogeneous systems always have at least one solution, namely, the trivial solution (the zero vector).

If system has at least one free variable then a homogeneous system will have infinite nontrivial solutions.

The solution to a nonhomogeneous system can be written as the sum of the solution to the homogeneous part and a particular solution which satisfies the nonhomogeneous part.

Example 1.10
The following matrix is in reduced echelon form. Write the solution to the corresponding system in vector form.

$$\left(\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 is free, x_1 and x_2 are basic.

$$\left.\begin{array}{l} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \end{array}\right\} \underline{x} = \begin{pmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{pmatrix}$$

$$\therefore \underline{x} = \underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{\text{Particular solution}} + \underbrace{x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}}_{\text{solution to homogeneous part}}$$

2 Elementary Matrices

The elementary row operations we perform whilst doing row reduction are as follows:

1. Swap rows.
2. Multiply a row by a non-zero constant.
3. Add a multiple of one row to another.

Each of these row operations has a corresponding elementary matrix for which when matrix multiplication is performed, results in that row operation.

> ***Definition***
>
> An ***elementary matrix*** is a matrix that can be obtained from the identity matrix by a single row operation.

If $\mathbf{E}$ is an elementary matrix obtained from a given row operation on the identity matrix, then the product, $\mathbf{EA}$ produces the same result as performing the row operation directly on the matrix, $\mathbf{A}$.

Example 2.1
Find the elementary matrices (3×3) that correspond with a) multiplying the second row by -3, b) swapping rows 1 and 3, c) adding 3 times the third row to the first. Verify the elementary matrices by left multiplying the following matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$$

a) $\underline{\underline{E}}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, b) $\underline{\underline{E}}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, c) $\underline{\underline{E}}_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\underline{\underline{E}}_1 \underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -6 & -3 & -9 \\ 1 & 1 & 0 \end{pmatrix} \checkmark$$

$$\underline{\underline{E}}_2 \underline{\underline{A}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix} \checkmark$$

$$\underline{\underline{E}}_3 \underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} \checkmark$$

We can also write a matrix as the product of elementary matrices since,

eg.

$$\underline{\underline{E}}_4 \underline{\underline{E}}_3 \underline{\underline{E}}_2 \underline{\underline{E}}_1 \underline{\underline{A}} = \underline{\underline{I}} \Longrightarrow \boxed{\underline{\underline{A}} = \underline{\underline{E}}_1^{-1} \underline{\underline{E}}_2^{-1} \underline{\underline{E}}_3^{-1} \underline{\underline{E}}_4^{-1}}$$

Remember! $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \frac{1}{|\underline{\underline{A}}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 2.2
Write the following matrix as a product of elementary matrices.

$$\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 3 & 8 \end{pmatrix}$$

$\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$, $R_1 \to -1R_1$, $\underline{\underline{E}}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\underline{\underline{E}}_1^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, $R_2 \to R_2 - 3R_1$, $\underline{\underline{E}}_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ | $\underline{\underline{E}}_2^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $R_2 \to \frac{R_2}{2}$, $\underline{\underline{E}}_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ | $\underline{\underline{E}}_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R_1 \to R_1 - 2R_2$, $\underline{\underline{E}}_4 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ | $\underline{\underline{E}}_4^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\underline{\underline{E}}_1^{-1}}\underbrace{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}}_{\underline{\underline{E}}_2^{-1}}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{\underline{\underline{E}}_3^{-1}}\underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}_{\underline{\underline{E}}_4^{-1}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 \\ 3 & 8 \end{pmatrix} = \underline{\underline{A}}$$

This technique leads us to a useful procedure of calculating inverse matrices.

2.1 Inverse of a Matrix

From the previous section we saw that we can write,

$$\underline{\underline{E}}_k \cdots \underline{\underline{E}}_2 \underline{\underline{E}}_1 \underline{\underline{A}} = \underline{\underline{I}}$$

Multiplying both sides <u>on the right</u> by $\mathbf{A}^{-1}$ gives,

$$\underline{\underline{E}}_K \cdots \underline{\underline{E}}_2 \underline{\underline{E}}_1 = \underline{\underline{A}}^{-1}$$

In other words:

> The product of the elementary matrices used to bring a matrix to reduced echelon form gives the inverse of the original matrix.

Applying this sequence of elementary matrices to both a matrix, $\mathbf{A}$, for which we wish to find the inverse, and the identity matrix returns the inverse of $\mathbf{A}^{-1}$.

$$\boxed{\underline{\underline{E}}_K \cdots \underline{\underline{E}}_2 \underline{\underline{E}}_1 \left[\underline{\underline{A}} \mid \underline{\underline{I}} \right] = \left[\underline{\underline{I}} \mid \underline{\underline{A}}^{-1} \right]}$$

So we can augment a matrix with the identity matrix, then do Gauss-Jordan elimination to obtain the identity matrix and the inverse.

Example 2.3
Find the inverse of the following matrix by Gauss-Jordan elimination.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c}{\underline{\underline{A}}} & \multicolumn{3}{c}{\underline{\underline{I}}} \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \\ \multicolumn{3}{c}{\underline{\underline{I}}} & \multicolumn{3}{c}{\underline{\underline{A}}^{-1}} \end{array}\right)$$

$$\therefore \underline{\underline{A}}^{-1} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}$$

$\ast$ If the reduced echelon form of a matrix is not the identity matrix then we cannot find an inverse for it.

3 LU-Factorisation

We have already seen how a matrix can be written as the product of elementary matrices.

We can also factor a matrix into a product,

$$\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$$

where **L** is a lower triangular matrix and **U** is an upper triangular matrix.

We call this *LU*-factorisation and can use it to solve linear systems.

In order to use this method we must first be able to factorise a matrix into the LU-components.

We achieve this using elementary matrices for row operations of type 3 (adding multiples of rows to each other).

Example 3.1
Find the LU-factorisation of the following matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{pmatrix}, \quad R_3 \to R_3 - 2R_1, \quad \underline{\underline{E}}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}, \quad R_3 \to R_3 + 4R_2, \quad \underline{\underline{E}}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

$$\underline{\underline{U}} \implies \underline{\underline{E}}_2\,\underline{\underline{E}}_1\,\underline{\underline{A}} = \underline{\underline{U}}$$

$$\therefore \underline{\underline{A}} = \underline{\underline{E}}_1^{-1}\,\underline{\underline{E}}_2^{-1}\,\underline{\underline{U}} = (\underline{\underline{E}}_1\,\underline{\underline{E}}_2)^{-1}\,\underline{\underline{U}}$$

$$= \underline{\underline{L}}\,\underline{\underline{U}}$$

where, $\underline{\underline{L}} = (\underline{\underline{E}}_1\,\underline{\underline{E}}_2)^{-1}$

eg: $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$, $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}$

$$E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}$$

$$= (E_1 E_2)^{-1}$$

(Alternative method for inverse of elementary matrices)

$$\underline{\underline{E}}_1 \underline{\underline{E}}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 4 & 1 \end{pmatrix}$$

$$\underline{\underline{A}}^{-1} = \frac{\operatorname{adj} \underline{\underline{A}}}{|\underline{\underline{A}}|}$$

"Transpose of cofactor matrix"

$$\operatorname{adj}(\underline{\underline{E}}_1 \underline{\underline{E}}_2) = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ -2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}, \quad \text{Also } |\underline{\underline{E}}_1 \underline{\underline{E}}_2| = 1$$

(triangular matrix)

$$\therefore \underline{\underline{L}} = \frac{\operatorname{adj}(\underline{\underline{E}}_1 \underline{\underline{E}}_2)}{|\underline{\underline{E}}_1 \underline{\underline{E}}_2|} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}$$

$$\therefore \underline{\underline{A}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}}_{\underline{\underline{L}}}\underbrace{\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}}_{\underline{\underline{U}}}$$

So this method allows us to simply calculate an echelon form (**U**) producing elementary matrices, then we can invert their product to find **L** to complete the factorisation.

Let us now use LU-factorisation to solve a system of linear equations.

For the system, $\mathbf{Ax} = \mathbf{b}$:

1. Obtain the LU-factorisation for the system.
2. Writing $\mathbf{y} = \mathbf{Ux}$ we can solve $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$.
3. Solve $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$.

Example 3.2

Solve the following system of linear equations using LU-factorisation.

$$
\begin{aligned}
x_1 - 3x_2 &= -5\\
x_2 + 3x_3 &= -1\\
2x_1 - 10x_2 + 2x_3 &= -20
\end{aligned}
$$

$$\underline{\underline{A}} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{pmatrix}$$

1) $\underline{\underline{L}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}, \quad \underline{\underline{U}} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}$

2) $\underline{y} = \underline{\underline{U}}\,\underline{x} \Longrightarrow \underline{\underline{L}}\,\underline{y} = \underline{\underline{A}}\,\underline{x} = \underline{b}$

$$\underset{\underline{\underline{L}}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}} \underset{\underline{y}}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} = \underset{\underline{b}}{\begin{pmatrix} -5 \\ -1 \\ -20 \end{pmatrix}}$$

"Forward substitution"

$y_1 = -5, \quad y_2 = -1, \quad y_3 = -20 - 2y_1 + 4y_2 = -14$

$$\underline{y} = \begin{pmatrix} -5 \\ -1 \\ -14 \end{pmatrix}$$

3) $\underline{\underline{U}}\,\underline{x} = \underline{y} \Longrightarrow \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -14 \end{pmatrix}$

"back substitution"

$x_3 = -1, \quad x_2 = -1 - 3x_3 = 2$

$x_1 = -5 + 3x_2 = 1$

$$\therefore \underline{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$